Better Analysis of GREEDY Binary Search Tree on Decomposable Sequences

Navin Goyal
Microsoft Research
Bangalore, India
navingo@microsoft.com

Manoj Gupta
IIT Gandhinagar
Gandhinagar, India
gmanoj@iitgn.ac.in

Abstract

In their seminal paper [Sleator and Tarjan, J.ACM, 1985], the authors conjectured that the *splay tree* is *dynamically optimal* binary search tree (BST). In spite of decades of intensive research, the problem remains open. Perhaps a more basic question, which has also attracted much attention, is if there exists *any* dynamically optimal BST algorithm. One such candidate is GREEDY which is a simple and intuitive BST algorithm [Lucas, Rutgers Tech. Report, 1988; Munro, ESA, 2000; Demaine, Harmon, Iacono, Kane and Patrascu, SODA, 2009]. [Demaine et al., SODA, 2009] showed a novel connection between a geometric problem and the binary search tree problem related to the above conjecture. However, there has been little progress in solving this geometric problem too.

Since dynamic optimality conjecture in its most general form remains elusive despite much effort, researchers have studied this problem on special sequences. Recently, [Chalermsook, Goswami, Kozma, Mehlhorn and Saranurak, FOCS, 2015] studied a type of sequences known as *k-decomposable sequences* in this context, where k parametrizes easiness of the sequence. Using tools from forbidden submatrix theory, they showed that GREEDY takes $n2^{O(k^2)}$ time on this sequence and explicitly raised the question of improving this bound.

In this paper, we show that GREEDY takes $O(n \log k)$ time on k-decomposable sequences. In contrast to the previous approach, ours is based on first principles. One of the main ingredients of our result is a new construction of a lower bound certificate on the performance of any algorithm. This certificate is constructed using the execution of GREEDY, and is more nuanced and possibly more flexible than the previous independent set certificate of Demaine et al. This result, which is applicable to all sequences, may be of independent interest and may lead to further progress in analyzing GREEDY on k-decomposable as well as general sequences.

1 Introduction

Binary search trees (BSTs) are a well-studied and fundamental data access model. We store keys from the universe $\{1,\ldots,n\}$ in a binary search tree and given an sequence of keys $(p_1,\ldots,p_n)$ we would like to access these keys (and possibly any associated data) using the tree. We would like to minimize the total cost of accessing the keys in this sequence, where the cost for one key search is the number of nodes touched for accessing that key. Various versions of this problem have been studied and some of them are very well understood, e.g., when the sequence of keys is generated according to some probability distribution with known access probabilities (and some additional restrictions) then optimal search trees are known (see the references in [14]). But what happens for general access sequences? The tree need not be static and can adapt to the access sequence. Such self-adjusting BST algorithms were considered by Sleator and Tarjan in

their seminal paper [14], where they introduced splay trees. These trees change dynamically via *rotations* after processing each access request. Sleator and Tarjan showed that the amortized cost of the splay tree is $O(\log n)$. They famously conjectured that splay trees have the much stronger and attractive property of *dynamic optimality*: for any (sufficiently long) sequence the total cost of the splay tree algorithm is within a constant factor of the total cost of the optimum *offline* binary search tree (i.e. the cost of the best BST algorithm that is given the access sequence in advance, and thus can decide how to change the tree based on this knowledge). Splay trees, by contrast, are *online*: the decision of how to change the trees must be based on the current access request only and cannot depend on the future requests. The above conjecture is called the *dynamic optimality conjecture*.

There has been much work on this conjecture, as well as on the more fundamental question of whether there exists *any* dynamically optimal BST algorithm (see [9] for a recent review). In the past decade progress was made on this latter question and BST algorithms with better competitive ratio were discovered: Tango trees [6] was the first $O(\log\log n)$-competitive BST; Multi-Splay trees [15] and Zipper trees [1] also have the same competitive ratio along with some additional properties. Analyses of these trees use lower bounds for the total cost to process a request sequence. Wilber [16] gave two different lower bounds. Wilber's first lower bound is used in [6, 15, 1] to obtain $O(\log\log n)$-competitive ratio for the respective trees. These techniques based on Wilber's bound have so far failed to give $o(\log\log n)$-competitiveness. Researchers have also proved conjectures implied by dynamic optimality; some of these can be interpreted as pertaining to *easy* sequences, e.g. for splay trees the amortized access cost is logarithmic in the distance in the key space to the previous key accessed (dynamic finger theorem) [3, 4], and amortized accesses cost is logarithmic in the temporal distance to the previous access to the current key (working set theorem) [14].

It turns out that even the problem of designing offline optimal BST algorithm has seen limited progress. Lucas [10] and Munro [11] designed a simple offline greedy BST algorithm and conjectured its cost to be close to the cost of optimum offline BST algorithm. Demaine et al. [6], using a novel geometric point of view of the problem, showed that surprisingly the offline greedy algorithm can in fact be turned into an online algorithm called GREEDY with only a constant factor loss in the competitive ratio (the ratio between the cost of the online algorithm to the cost of the offline optimum for the worst case access sequence). Thus the conjecture about the the offline greedy would imply that GREEDY is dynamically optimal. The geometric view of the BST problem mentioned above leads to a completely different looking clean problem about point sets in the plane. This raises the possibility of new lines of attack that might be harder to conceive in the BST view. Unfortunately, so far success has been limited even with the geometric view. The current state of the art by Fox [8] shows that GREEDY takes $O(n\log n)$ time for any arbitrary sequence $\mathscr{X}$.

Given this state of affairs, it has been suggested by several researchers to study the problem for easy sequences (see [2] and references therein). Just as for splay trees, the question arises about the performance of GREEDY on *easy* sequences; in particular, does the geometric approach help? In this context, Chalermsook et al. [2] initiated the study of GREEDY on *decomposable sequences* and brought to bear techniques from *forbidden submatrix theory* to this problem and some other problems. (Some of the tools from forbidden submatrix theory have been used previously by Pettie [12, 13] to achieve better bounds for splay trees on deque sequences and a new proof of the sequential access theorem for splay trees.) They showed that GREEDY takes $n2^{O(k^2)}$ time on k-decomposable sequences. Furthermore, they showed optimal offline cost for k-decomposable sequences is $\Theta(n\log k)$. We quote from their paper:

"A question directly related to our work is to close the gap between $\text{OPT} = O(n\log k)$ and $n2^{O(k^2)}$ by GREEDY on k-decomposable sequences (when $k = \omega(1)$). Matching the optimum (if at all possible) likely requires novel techniques: splay is not even known to be linear on preorder sequences with preprocessing, and with forbidden-submatrix-arguments it seems impossible to obtain bounds beyond $O(nk)$."

Though the authors mention that forbidden submatrix theory may give an $O(nk)$ bound, it is not clear how to achieve this goal. We solve the above open problem by showing

Theorem 1.1. GREEDY *takes* $O(n\log k)$ *time on* k*-decomposable sequences* (*with preprocessing*[1]).

Our approach is based on first principles and does not use tools from forbidden submatrix theory. We carefully analyze execution of GREEDY on k-decomposable sequences and discover some new structural properties of GREEDY. Our proof also uses the aforementioned general technique of constructing lower bound certificates on the cost of the optimum and relating this lower bound to the cost of the algorithm being analyzed. One such lower bound, *independent set* lower bound was provided by Demaine et al. [6]. It was, however, not clear how to relate it, or its close relatives, to the cost of GREEDY. Our lower bound certificate is derived from the execution of GREEDY; it builds upon the ideas of independent set lower bound, but provides a more nuanced and possibly more flexible certificate. Our certificate construction works for general sequences and not just for k-decomposable sequences. We are hopeful that our approach will lead to further progress in understanding the performance of GREEDY on k-decomposable and general sequences.

2 The Geometric Problem

Let $[n] = \{1, 2, \dots, n\}$ denote the set of keys. Let $(p_1, p_2, \dots, p_n)$ denote a permutation on n keys, i.e., $p_i \neq p_j$ for $i \neq j$. We can represent this permutation by a set of n points in the plane: Key p_i is represented by the point (p_i, i). For a point p, let $p.x$ denote its x-coordinate and let $p.y$ denote its y-coordinate. We will denote sets of points obtained from permutations of keys in this way by $\mathscr{X}$. There is exactly one point of $\mathscr{X}$ on the line $x = i$, for $i \in [n]$; and there is exactly one point from $\mathscr{X}$ on the line $y = i$, for $i \in [n]$. Clearly, there is a one-to-one correspondence between the permutations and sets of points as defined above. For the most part we will use the point set view.

In our paper, the positive x-axis (representing key space) moves from left to right and the positive y-axis (representing time) moves from *top to bottom* (this latter convention is different from most previous papers in this area). For a pair of points p and q not on the same horizontal or vertical line, the (closed) axis-aligned rectangle formed by p and q is denoted by ${}^{q}\square_{p}$ if $q.y < p.y$ and $q.x < p.x$ and ${}_{p}\square^{q}$ if $q.y < p.y$ and $p.x < q.x$.

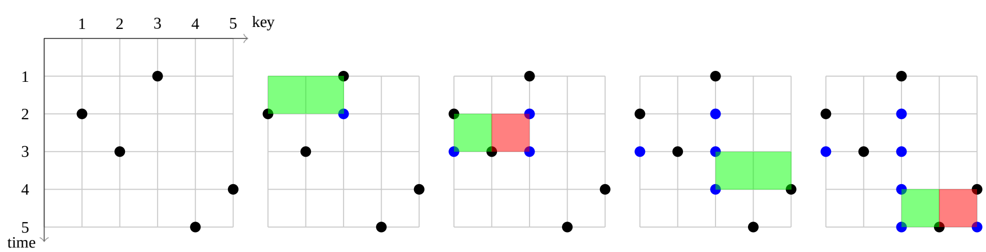


Figure 1: The first picture shows the point set X={3,1,2,5,4}. In the remainder of the paper we do not show the x-axis and the y-axis (along with the first row and column of the grid). The execution of GREEDY is shown from the second picture onwards.

[1]The preprocessing step is the same as in [6, 2], that is, insert all elements into a *split tree* before processing any requests. Preprocessing is independent of the access sequence. See [6, 2] for more details

Definition 2.1. *A pair of points (p,q) is said to be arborally satisfied with respect to a point set P, if (1) p and q lie on the same horizontal or vertical line or, (2) $\exists r \in P \backslash \{p,q\}$ such that r lies in the interior or boundary of ${}^{q}\square_{p}$ (or ${}_{p}\square^{q}$).*

We say that the rectangle ${}^{q}\square_{p}$ (or ${}_{p}\square^{q}$) is arborally satisfied if condition (2) holds, otherwise it is arborally unsatisfied. Consider the following problem: *Given a point set $\mathscr{X}$, find a minimum cardinality point set $\mathscr{Y}$ such that each pair of point in the set $\mathscr{X} \cup \mathscr{Y}$ is arborally satisfied.*

If each pair of points in $\mathscr{X} \cup \mathscr{Y}$ is arborally satisfied then we say that the set $\mathscr{X} \cup \mathscr{Y}$ is an arborally satisfied set otherwise it is not. In their remarkable paper, Demaine et al. [5] formulated the above problem. They showed a novel connection between this *geometric* problem and binary search tree (BST) problem, and designed a simple algorithm, henceforth called GREEDY, for the above geometric problem. Let $\mathscr{G}$ be the set of points added by this algorithm described as follows:

Sweep the point set $\mathscr{X}$ with a horizontal line by increasing the y-coordinates. Let the point p be processed at time $p.y$. At time $p.y$, place the minimal number of points on line $y = p.y$ to satisfy the rectangle with p as one endpoint and other endpoint in $\mathscr{X} \cup \mathscr{G}$ having their y-coordinate less than $p.y$. This minimal set of points M_p is uniquely defined: for any arborally unsatisfied rectangle formed with p in one corner, add a point at the other corner at $y = p.y$ in $\mathscr{G}$. Please see Figure 1 for the execution of GREEDY.
One can show that the $\mathscr{X} \cup \mathscr{G}$ is an arborally satisfied set; see [5] for details.

3 Overview

We give a brief overview of our techniques in this section. Our goal is to prove $|\mathscr{G}| = O(n \log k)$ which immediately implies Theorem 1.1. The starting point of our approach was an attempt to construct an independent set of rectangles of original points certifying a lower bound on the number of points that must be marked. We attempt to construct such a certificate by analyzing the execution of GREEDY. Our final certificate will not be an independent set however.

In Sec. 6, for each point in $\mathscr{G}$ we associate a tuple of points (which we also think of as a rectangle) from $\mathscr{X}$ using a map called PAIR$(\cdot)$. At a high level, this can be thought of looking for a reason for why the point in $\mathscr{G}$ was marked by GREEDY. We partition PAIR$(\mathscr{G})$ into two sets called $\zig$ (pronounced zig) and $\zag$ (pronounced zag). The visual notation $\zig$ and $\zag$ depicts how PAIR(p) is located w.r.t. $p \in \mathscr{G}$. In Sections 7 we show that $|\zig| = O(nk)$ and in Sec. 8 we improve it to $|\zig| = O(n \log k)$; this has a relatively short proof and uses properties of $\zig$ and k-decomposable sequences.

We then show $|\zag| = O(nk)$. The proof of this is in two parts. First we construct the partition $\zag =$ GOOD$(\zag) \cup$ BAD$(\zag)$ (Sec. 9.1). The set GOOD$(\zag)$ consists of rectangles from $\zag$ that do not have any original points in their interior (thus this set can presumably be quite different from an independent set). In Sections 9.2 and 9.3, we analyze GOOD$(\zag)$ and show that it provides a lower bound certificate for $|\text{OPT}(\mathscr{X})|$ similar to the independent set certificate of Demaine et al. [5]: $|\text{GOOD}(\zag)|/2 + |\mathscr{X}| \leqslant |\text{OPT}(\mathscr{X}) \cup \mathscr{X}|$. We remark that this result holds for all $\mathscr{X}$ and not just for k-decomposable sequences, and hence may be of use in future work on the general problem. Chalermsook et al. have proven $|\mathscr{X} \cup \text{OPT}(\mathscr{X})| \leqslant O(n \log k)$ for k-decomposable sequences, which implies $|\text{GOOD}(\zag)| = O(n \log k)$. Finally, in Sec. 10 we show $|\text{BAD}(\zag)| = O(nk)$ and then in Sec. 11 improve it to $|\text{BAD}(\zag)| = O(n \log k)$. For this, we use some structural properties of BAD$(\zag)$, $\zig$, and k-decomposable sequences. The above results together imply our desired bound $|\mathscr{G}| = |\text{PAIR}(\mathscr{G})| = O(n \log k)$ (Sec. 12).

4 Basic Properties of GREEDY

In all our diagrams, a point in $\mathscr{X}$ is denoted by a black circle and a point in $\mathscr{G}$ is denoted by a blue circle. A point in $(\mathscr{X} \cup \mathscr{G})$ is denoted by a gray circle. A point in $\mathscr{X}$ will be called an *original* point and a point in $\mathscr{G}$ will be called a *marked* point. When we refer to a point without specifying whether its marked or original, then it is a point from $(\mathscr{X} \cup \mathscr{G})$. We use following notation:

1. p is *above* q (or q is below p) or $\underset{q}{\overset{p}{\updownarrow}}$ if $p.y < q.y$ and $p.x = q.x$.
2. p is to the *left* of q (or q is to the right of p) or $p \leftrightarrow q$ if $p.x < q.x$ and $p.y = q.y$.
3. q is to the *south-east* of p (or p is to the north-west of q) or ${}^{p}\!\nwarrow\!\!\searrow_{q}$ if $p.y < q.y$ and $p.x < q.x$.
4. q is to the *north-east* of p (or p is to the south-west of q) or ${}_{p}\!\swarrow\!\!\nearrow^{q}$ if $p.y > q.y$ and $p.x < q.x$.
5. $({}_{p}\square^{q})^{\circ}$ denotes the interior of ${}_{p}\square^{q}$.
6. While processing $p \in \mathscr{X}$, GREEDY may put marked points on the line $y = p.y$ (there is no other original point on this line as $\mathscr{X}$ comes from a permutation sequence). For any such marked point q we define its original point $\mathrm{OP}(q)$ to be p. We also set $\mathrm{OP}(p) := p$.
7. For $q \in \mathscr{X} \cup \mathscr{G}$, define $\mathrm{UP}(q) := q$ if $q \in \mathscr{X}$, and $\mathrm{UP}(q) := p$ if $q \in \mathscr{G}$, where $p \in \mathscr{X}$ is the unique original point above q (see the discussion before Observation 4.1 below).

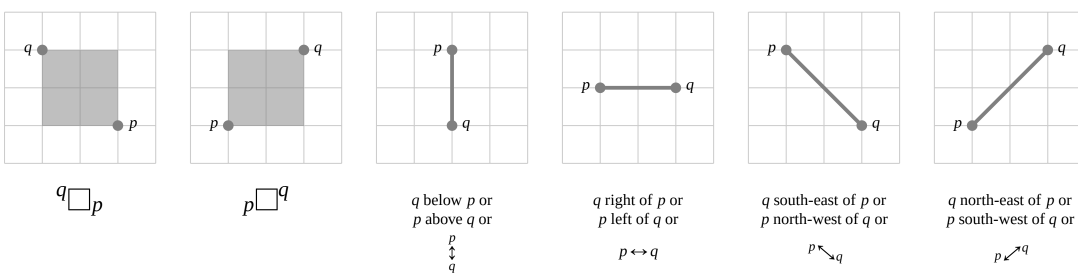


Figure 2: Basic Notations

We now show some basic properties and lemmas related to the execution of GREEDY.

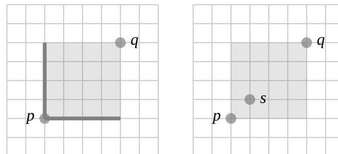
While preprocessing p, GREEDY adds a marked point at the bottom right (bottom left) corner of rectangle ${}_{p}\square^{q}$ (${}^{q}\square_{p}$) only if it is an arborally unsatisfied rectangle. This implies that whenever GREEDY puts a marked point there exists another point (marked or original) above it. Using this property, we claim that the top point on the line $x = i$ $(1 \leqslant i \leqslant n)$ must be an original point, i.e., a point from $\mathscr{X}$. The following observation follows:

Observation 4.1. *For any point $p \in \mathscr{X}$, GREEDY does not put any marked point above p.*

We now prove some lemmas regarding the execution of GREEDY:

Lemma 4.2. *Consider a rectangle ${}_{p}\square^{q}$ where $p, q \in (\mathscr{X} \cup \mathscr{G})$. Then there exists a point $r \in (\mathscr{X} \cup \mathscr{G}) \setminus \{p, q\}$ such that (1) $r \in {}_{p}\square^{q}$, and (2) $\underset{p}{\overset{r}{\updownarrow}}$ or $p \leftrightarrow r$.*

Remark 4.3. *This lemma is a special case of Observation 2.1 in [5]. The lemma is true for any point set $\mathscr{Y}$ for which $(\mathscr{X} \cup \mathscr{Y})$ is an arborally satisfied set.*



(i) (ii)

Figure 3: (1) Lemma 4.2 states that there exists a point in $(\mathscr{X} \cup \mathscr{G})\backslash\{p,q\}$ on the thick lines adjacent to p. (2) Illustration of the case argued in the proof where s is the closest point to p in ${}_p\square^q$.

Proof. Since $(\mathscr{X} \cup \mathscr{G})$ is an arborally satisfied set, there exists another point from $(\mathscr{X} \cup \mathscr{G})\backslash\{p,q\}$ in ${}_p\square^q$. Let s be the closest point to p in ${}_p\square^q$ (closest in Euclidean distance) (See Figure 3(ii)). If $\overset{s}{\underset{p}{\updownarrow}}$ or $p \leftrightarrow s$, then we are done. Else, look at the rectangle ${}_p\square^s$. Since s is the closest point to p, ${}_p\square^s$ does not contain any other point from $(\mathscr{X} \cup \mathscr{G})\backslash\{p,s\}$. This implies that ${}_p\square^s$ is not arborally satisfied. This leads to a contradiction as $(\mathscr{X} \cup \mathscr{G})$ is an arborally satisfied set. ☐

Similarly, we can also prove a symmetric version of the above lemma:

Lemma 4.4. *Consider a rectangle ${}^q\square_p$ where $p,q \in (\mathscr{X} \cup \mathscr{G})$. Then there exists a point $r \in (\mathscr{X} \cup \mathscr{G})\backslash\{p,q\}$ such that (1) $r \in {}^q\square_p$, and (2) $\overset{r}{\underset{p}{\updownarrow}}$ or $r \leftrightarrow p$.*

We now move on to another important property of GREEDY.

Definition 4.5. *A point r* hides *q from p where $p,q,r \in \mathscr{X} \cup \mathscr{G}$ and ${}^q\!\nwarrow\!\!\searrow_p$, if either (1) $q \leftrightarrow r$ and $r \in {}^q\square_p$, or (2) $r \in ({}^q\square_p)^\circ$.*

For p,q such that ${}_p\!\swarrow\!\!\nearrow^q$ the definition is symmetric.

In other words, r hides q from p, if it's different from p,q and lies in the union of the "top-line" and the interior of ${}^q\square_p$ (or ${}_p\square^q$).

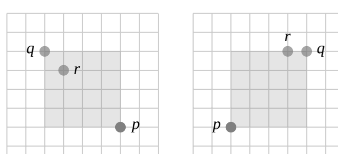


Figure 4: r hides q from $p \in \mathscr{X}$

Lemma 4.6. *Let r hide q from p where $p,q,r \in \mathscr{X} \cup \mathscr{G}$ and ${}^q\!\nwarrow\!\!\searrow_p$. Assume that there exists a point s below q such that ${}_s\!\swarrow\!\!\nearrow^r$. Moreover, let s be the first point below q with this property. Then ${}_{\mathrm{OP}(s)}\!\swarrow\!\!\nearrow^r$.*

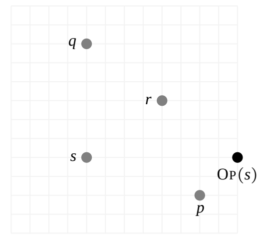


Figure 5: Illustration of bad case in the proof of Lemma 4.6 when ${}^{r}\nwarrow\!\!\searrow_{\mathrm{OP}(s)}$

Proof. Assume that r lies in $({}^{q}\square_{p})^{\circ}$ (the argument below applies even if $q \leftrightarrow r$). By Observation 4.1, GREEDY does not put any marked point above any original point. So $\mathrm{OP}(s)$ cannot lie below r. Assume then for contradiction that ${}^{r}\nwarrow\!\!\searrow_{\mathrm{OP}(s)}$ (See Figure 5). Since GREEDY puts s while processing $\mathrm{OP}(s)$, it must have encountered an unsatisfied rectangle ${}^{t}\square_{\mathrm{OP}(s)}$ such that t is the first point above s. We claim that t cannot lie to the north-west or left of r as then ${}^{t}\square_{\mathrm{OP}(s)}$ is arborally satisfied by r. This implies that ${}_{t}\swarrow\!\!\nearrow^{r}$. But then t is the first point below q such that ${}_{t}\swarrow\!\!\nearrow^{r}$, which contradicts the assumption of the lemma. So our assumption that ${}^{r}\nwarrow\!\!\searrow_{\mathrm{OP}(s)}$ must be false. $\square$

We also state the symmetric version of the above lemma:

Lemma 4.7. *Let r hide q from p where $p,q,r \in \mathscr{X} \cup \mathscr{G}$ and ${}_{p}\swarrow\!\!\nearrow^{q}$. Assume that there exists a point s below q such that ${}^{r}\nwarrow\!\!\searrow_{s}$. Moreover, let s be the first point below q with this property. Then ${}^{r}\nwarrow\!\!\searrow_{\mathrm{OP}(s)}$.*

5 Decomposable Sequences

Given a permutation of the keys $(p_1, p_2, \ldots, p_n)$, represented by point set $\mathscr{X}$ in the plane as described above, we call a set $[i, j] := \{i, i+1, \ldots, j\}$ a block of $\mathscr{X}$ if $\{p_i, p_{i+1}, \ldots, p_j\} = \{c, c+1, \ldots, d\}$ for some $c, d \in [n]$. In words, a block represents a contiguous time interval that is mapped to a contiguous key interval by the permutation. We say that $\mathscr{X}$ is decomposable into k-blocks if there exist disjoint blocks $[a_1, b_1], \ldots, [a_k, b_k]$ such that $\cup_{\ell}[a_\ell, b_\ell] \cap \mathbb{N} = [n]$.

We can recursively decompose $\mathscr{X}$ till singleton blocks are obtained. This recursive decomposition can be naturally represented as a rooted tree where each node represents a block. At the root of the tree is the block $\mathscr{X} = (p_1, p_2, \ldots, p_n)$. Let us call this tree a *decomposition tree* of $\mathscr{X}$. We say that $\mathscr{X}$ is *k-decomposable* if there exists a decomposition tree $\mathrm{TREEDECOMPOSITION}(\mathscr{X})$ such that the number of children of each internal node in this tree is at most k.

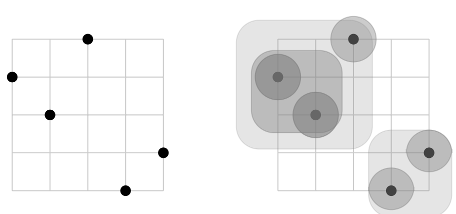


Figure 6: The point set $\mathscr{X}$=$\{3,1,2,5,4\}$ and its decomposition

Let $\text{PARENT}(B)$ denote the parent of B in $\text{TREEDECOMPOSITION}(\mathscr{X})$, and let $\text{SIBLING}(B)$ denote the set of children of $\text{PARENT}(B)$ except B. Let $\text{TOP}(B) := a$ if $a \in B$ and $a.y \leqslant p.y$ for all $p \in B$. In words, $\text{TOP}(B)$ denotes the first (in time) key in block B. Note that a key a can be the first key of multiple blocks (at different levels). In Fig. 6 , $\text{TOP}(3,1,2,5,4) = \text{TOP}(3,1,2) = \text{TOP}(3) = 3$. Let $\text{TOPBLOCK}(a)$ be the block B that is closest to the root and satisfies $\text{TOP}(B) = a$. In Fig. 6, $\text{TOPBLOCK}(3) = (3,1,2,5,4)$.

In the rest of this paper we deal with k-decomposable permutations $\mathscr{X}$, or more precisely, with point sets $\mathscr{X}$ representing such permutations. We fix some $\text{TREEDECOMPOSITION}(\mathscr{X})$ such that every internal node has at most k children. Henceforth when we talk about blocks, it will be with respect to this fixed $\text{TREEDECOMPOSITION}(\mathscr{X})$.

For a block B, let $\text{BOX}(B) := \{r \mid \exists p,q \in B \text{ s.t. } r.x = p.x \text{ and } r.y = q.y\}$. In words, $\text{BOX}(B)$ contains those points r in the plane such that both the horizontal and vertical lines passing through r have at least one point from B.

Define $\text{UPPERBOX}(B) := \{r \mid r.y < \text{TOP}(B).y \text{ and } \exists p \in B \text{ such that } p.x = r.x\}$. In words, $\text{UPPERBOX}(B)$ is the set of points in the plane that come before all points in B in time, but share the key with some point in B. Please see the Figure 7.

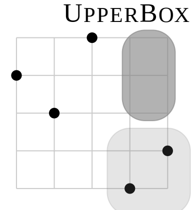


Figure 7: $\text{BOX}(B)$ and $\text{UPPERBOX}(B)$ contains the set of all the points in the two region shown in the figure.

By definition, a block represents a contiguous key interval. Hence, there exists no point $q \in \mathscr{X}$ in $\text{UPPERBOX}(B)$. Also by Observation 4.1, for any point $q \in \mathscr{X}$, GREEDY does not put any point above q, so we have

Lemma 5.1. *There is no point from* $(\mathscr{X} \cup \mathscr{G})$ *in* $\text{UPPERBOX}(B)$.

Remark 5.2. *For brevity, our definitions and theorem and lemma statements do not explicitly quantify over* $\mathscr{X}$*, but there is always an underlying "For a permutation sequence* $\mathscr{X}$*". In Sections 6 and 9, this quantification is over all* $\mathscr{X}$ *of size n; and in Sections 7, 10, and 12, it is over all k-decomposable* $\mathscr{X}$ *of size n.*

6 Pairs

We define a map from $\mathscr{G}$ to pairs of points in $\mathscr{X}$.

Definition 6.1. *Let* $p \in \mathscr{G}$*. The map* $\text{PAIR} : \mathscr{G} \to \mathscr{X}^2$ *is defined as follows: Let q be a the first point above p, then* $\text{PAIR}(p) := (\text{OP}(p), \text{OP}(q))$.

Note that if $\textsc{Pair}(p) = (\textsc{Op}(p), \textsc{Op}(q))$, then $\textsc{Op}(p).y > \textsc{Op}(q).y$.

Our goal is to upper bound $|\mathscr{G}|$. We do this by connecting $|\mathscr{G}|$ to the set of all *pairs* $\textsc{Pair}(\mathscr{G}) := \{\textsc{Pair}(p) \mid p \in \mathscr{G}\}$, partitioning the set $\textsc{Pair}(\mathscr{G})$ into two sets $^{\cdot}.{:}$ and $.{:}^{\cdot}$, and then bounding $|^{\cdot}.{:}|$ and $|.{:}^{\cdot}|$ from above.

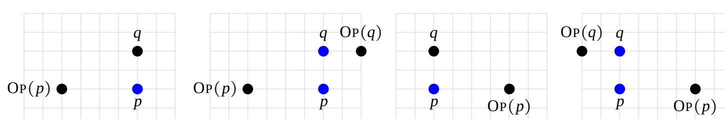
Definition 6.2 ($^{\cdot}.{:}$)**.** *Subset* $^{\cdot}.{:}$ *of* $\textsc{Pair}(\mathscr{G})$*, pronounced* zig*, is defined as follows. For* $p \in \mathscr{G}$*, let* q *be the first point above* p *and* $\textsc{Pair}(p) = (\textsc{Op}(p), \textsc{Op}(q))$*. We say that* $\textsc{Pair}(p) \in {}^{\cdot}.{:}$ *if (1)* $\textsc{Op}(p) \leftrightarrow p$ *and* ${}^{\textsc{Op}(q)}\nwarrow\!\!\searrow_{\textsc{Op}(p)}$*, or its symmetric version (2)* $p \leftrightarrow \textsc{Op}(p)$ *and* ${}_{\textsc{Op}(p)}\swarrow\!\!\nearrow^{\textsc{Op}(q)}$.

(i) (ii)

Figure 8: $\textsc{Pair}(p) \in {}^{\cdot}.{:}$. In Figure (i) $\textsc{Pair}(p) \in \mathrm{R}(^{\cdot}.{:})$ and in Figure (ii) $\textsc{Pair}(p) \in \mathrm{L}(^{\cdot}.{:})$

Note that the symbol $^{\cdot}.{:}$ mimics Fig. 8(i). Let $\mathrm{R}(^{\cdot}.{:}) = \{\textsc{Pair}(p) \mid \textsc{Pair}(p) \in {}^{\cdot}.{:} \text{ and } \textsc{Op}(p) \leftrightarrow p\}$. Similarly, $\mathrm{L}(^{\cdot}.{:}) = \{\textsc{Pair}(p) \mid \textsc{Pair}(p) \in {}^{\cdot}.{:} \text{ and } p \leftrightarrow \textsc{Op}(p)\}$.

Definition 6.3 ($.{:}^{\cdot}$)**.** *Subset* $.{:}^{\cdot}$ *of* $\textsc{Pair}(\mathscr{G})$*, pronounced* zag*, is defined as follows. For* $p \in \mathscr{G}$*, let* q *be the first point above* p *and* $\textsc{Pair}(p) = (\textsc{Op}(p), \textsc{Op}(q))$*. We say that* $\textsc{Pair}(p) \in .{:}^{\cdot}$ *if (1)* $\textsc{Op}(p) \leftrightarrow p$ *and (* $\begin{smallmatrix}\textsc{Op}(q)\\ \updownarrow\\ p\end{smallmatrix}$ *or* ${}_{p}\swarrow\!\!\nearrow^{\textsc{Op}(q)}$*), or its symmetric version (2)* $p \leftrightarrow \textsc{Op}(p)$ *and (* $\begin{smallmatrix}\textsc{Op}(q)\\ \updownarrow\\ p\end{smallmatrix}$ *or* ${}^{\textsc{Op}(q)}\nwarrow\!\!\searrow_{p}$*).*



(i) (ii) (iii) (iv)

Figure 9: $\textsc{Pair}(p) \in .{:}^{\cdot}$.

Note that the symbol $.{:}^{\cdot}$ mimics Fig. 9(ii).

The following observation follows from the definition of $.{:}^{\cdot}$:

Observation 6.4. *If* $\textsc{Pair}(p) = (\textsc{Op}(p), \textsc{Op}(q))$ *and* $\textsc{Pair}(p) \in .{:}^{\cdot}$*, then* p *lies to the left (right) of* $\textsc{Op}(p)$ *in* ${}^{\textsc{Op}(q)}\square_{\textsc{Op}(p)}$ $({}_{\textsc{Op}(p)}\square^{\textsc{Op}(q)})$.

Henceforth, we will abuse notation and use $\textsc{Pair}(p_1)$ (with $\textsc{Pair}(p_1) \in .{:}^{\cdot}$) as an ordered pair (p, q) as well as ${}^{q}\square_{p}$ (or ${}_{p}\square^{q}$). This makes sense as there is a one-to-one correspondence between the tuples from $\mathscr{X}$ and rectangles with endpoints in $\mathscr{X}$ as $\mathscr{X}$ comes from a permutation sequence.

We now show some properties of $\textsc{Pair}(\cdot)$. Let $\mathrm{R}_{\mathscr{G}} := \{p \in \mathscr{G} \mid \textsc{Op}(p) \leftrightarrow p\}$; define $\mathrm{L}_{\mathscr{G}}$ similarly.

Lemma 6.5. *Let $p_1, p_2 \in \mathrm{R}_{\mathscr{G}}$. If $p_1 \neq p_2$ then $\textsc{Pair}(p_1) \neq \textsc{Pair}(p_2)$.*

Proof. Assume for contradiction that $\textsc{Pair}(p_1) = \textsc{Pair}(p_2)$. Then p_1 and p_2 are on the same horizontal line. Assume w.l.o.g. that $p_1 \leftrightarrow p_2$ and $\textsc{Op}(p_1) = \textsc{Op}(p_2) = p$. Assume that while processing $p \in \mathscr{X}$, Greedy marks point p_1 and p_2 due to unsatisfied rectangle ${}_{p}\square^{q_1}$ and ${}_{p}\square^{q_2}$. This implies that ${}^{q_1}\nwarrow\!\!\searrow_{q_2}$, as otherwise ${}_{p}\square^{q_2}$ is satisfied by q_1 (since $p_1 \leftrightarrow p_2$, $q_1 \leftrightarrow q_2$). Also note that $\textsc{Pair}(p_1) = (p, \textsc{Op}(q_1))$ and $\textsc{Pair}(p_2) = (p, \textsc{Op}(q_2))$. Since $q_1.y \neq q_2.y$, $\textsc{Op}(q_1) \neq \textsc{Op}(q_2)$. So $\textsc{Pair}(p_1) \neq \textsc{Pair}(p_2)$, contradicting our assumption. $\square$

This implies that $|\mathrm{R}_{\mathscr{G}}| \leqslant |\textsc{Pair}(\mathscr{G})|$. By symmetry, we also have $|\mathrm{L}_{\mathscr{G}}| \leqslant |\textsc{Pair}(\mathscr{G})|$. This gives

Corollary 6.6. $|\mathscr{G}| = |L_{\mathscr{G}}| + |R_{\mathscr{G}}| \leqslant 2|\textsc{Pair}(\mathscr{G})|$.

We now show that for any $p \in \mathscr{G}$, $\textsc{Pair}(p)$ is either in •.: or in .:• (thus the third possibility of pairs in Fig. 10, or its symmetric version, does not arise):

Lemma 6.7. *For $p \in \mathscr{G}$, either $\textsc{Pair}(p) \in$ •.: or $\textsc{Pair}(p) \in$.:•.*

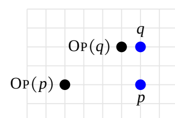


Figure 10: Illustration of the bad case which arises in case (3) of the proof of Lemma 6.7

Proof. W.l.o.g., let $\textsc{Op}(p) \leftrightarrow p$ and let q be the first point above p. So, $\textsc{Pair}(p) = (\textsc{Op}(p), \textsc{Op}(q))$. We have $\textsc{Pair}(p) \notin$ •.: and $\textsc{Pair}(p) \notin$.:• only if ${}_{\textsc{Op}(p)}\swarrow\!\!\nearrow^{\textsc{Op}(q)}$ and ${}^{\textsc{Op}(q)}\nwarrow\!\!\searrow_{p}$ (see Fig. 10). So, $\textsc{Op}(q)$ hides q from $\textsc{Op}(p)$ (as $\textsc{Op}(q) \leftrightarrow q$ in ${}_{\textsc{Op}(p)}\square^{q}$) and p is the first point below q such that ${}^{\textsc{Op}(q)}\nwarrow\!\!\searrow_{p}$ (this follows from the definition of $\textsc{Pair}(p)$ where we said that q is the first point above p). Then by Lemma 4.7, ${}^{\textsc{Op}(q)}\nwarrow\!\!\searrow_{\textsc{Op}(p)}$. This contradicts our assumption that ${}_{\textsc{Op}(p)}\swarrow\!\!\nearrow^{\textsc{Op}(q)}$. So our assumption on the position of $\textsc{Op}(q)$ must be false. $\square$

This shows that $\textsc{Pair}(\mathscr{G})$ is partitioned by •.: and .:•:

Corollary 6.8. |•.:| + |.:•| $= |\textsc{Pair}(\mathscr{G})|$.

We now show some properties of $\textsc{Pair}(\cdot)$ when the sequence $\mathscr{X}$ is decomposable. The following property is independent of whether $\textsc{Pair}(\cdot)$ is in •.: or .:•.

Lemma 6.9. *Let $p_1 \in \mathscr{G}$ and $p, q \in \mathscr{X}$ and $\textsc{Pair}(p_1) = (p, q)$. If $p \in B$ but $p \neq \textsc{Top}(B)$, then $q \in B$.*

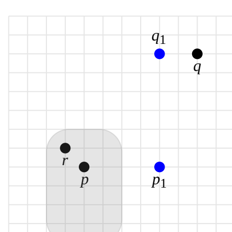


Figure 11: Lemma 6.9 shows that the scenario depicted in the figure cannot occur, i.e., $p \neq \textsc{Top}(B)$, $\textsc{Pair}(p_1) = (p, q)$ and $q \notin B$

Proof. Assume for contradiction that $\textsc{Pair}(p_1) = (p,q)$ such that $q \notin B$ (see Fig. 11 for an illustration). W.l.o.g., let $p \leftrightarrow p_1$. Let ${}_p\square^{q_1}$ be the unsatisfied rectangle, that made GREEDY mark p_1 while processing $p \in \mathscr{X}$. Note that $\textsc{Op}(q_1) = q$ can lie to the right or left of q_1 or it can be same as q_1 (though in Fig. 11 it lies to the right of q_1).

Let $\textsc{Top}(B) = r$. Note that r cannot lie above p (as points from $\mathscr{X}$ come from a permutation sequence) or to the north-east of p as then ${}_p\square^{q_1}$ is already arborally satisfied due to r. So ${}^{r}\!\nwarrow\!\!\searrow_{p}$ as shown in Fig. 11. By Lemma 5.1, no point lies in $\textsc{UpperBox}(B)$. This implies that $\textsc{Greedy}$ should find even ${}_r\square^{q_1}$ unsatisfied. In that case, $\textsc{Greedy}$ should put another marked point, say r' to the right of r below q_1. However, this contradicts our deduction that while processing p, $\textsc{Greedy}$ found ${}_p\square^{q_1}$ unsatisfied (because then ${}_p\square^{q_1}$ is already satisfied by r'). So we arrive at a contradiction and our assumption that $q \notin B$ must be false.

$\square$

In words, the above lemma says that for a block B all the points marked by $\textsc{Greedy}$ for $p \in B$ have pairs *local* to B if $\textsc{Top}(B) \neq p$. However, if $\textsc{Top}(B) = p$, the above claim does not hold. In Section 5, we saw that there can be many blocks for which $p = \textsc{Top}(\cdot)$. Let $\textsc{TopBlock}(p) = B$. For such p, B, we now show that all points marked by $\textsc{Greedy}$ will have *non-local* pairs.

Lemma 6.10. *Let $p, q \in \mathscr{X}$, $p_1 \in \mathscr{G}$ and $\textsc{TopBlock}(p) = B$. If $\textsc{Pair}(p_1) = (p,q)$, then $q \in B'$ where $B' \in \textsc{Sibling}(B)$.*

Proof. By definition, if $\textsc{Pair}(p_1) = (p,q)$ then $q.y < p.y$. Since p is the first original point in B with the least y-coordinate, $q \notin B$. Since $\textsc{TopBlock}(p) = B$, $p \neq \textsc{Top}(\textsc{Parent}(B))$. Hence, by Lemma 6.9, $q \in \textsc{Parent}(B)$. This implies that $q \in B'$ such that $B' \in \textsc{Sibling}(B)$. $\square$

7 Upper bounding $|\cdot\!:|$

In this section, we prove

Theorem 7.1. $|\cdot\!:| \leqslant 2n(k-1)$.

Consider a point $p \in \mathscr{X}$. In Section 6, we proved Lemma 6.10 which says that if $\textsc{Pair}(p_1) = (p, q_1)$ and $\textsc{TopBlock}(p) = B$, then $q_1 \in B_1$ where $B_1 \in \textsc{Sibling}(B)$. We will now prove a certain strengthening of this lemma. Let $\mathrm{R}(\cdot\!:)_p = \{\textsc{Pair}(p_i) \in \mathrm{R}(\cdot\!:) \mid \textsc{Pair}(p_i) = (p, q_i) \text{ for some } q_i \in \mathscr{X}\}$. In the next section, we prove Lemma 7.3 which essentially says that if $\textsc{Pair}(p_1), \dots, \textsc{Pair}(p_\ell) \in \mathrm{R}(\cdot\!:)_p$ with $\textsc{Pair}(p_i) = (p, q_i)$, then blocks B_i containing q_i are pairwise distinct. By the k-decomposability of $\mathscr{X}$, there are at most $k-1$ siblings of B, hence we have $\ell \leqslant k-1$, which gives $|\mathrm{R}(\cdot\!:)_p| \leqslant k-1$. The same reasoning holds for $\mathrm{L}(\cdot\!:)_p$. And this immediately gives the bound on $|\cdot\!:|$.

The following lemma will be used in proving Lemma 7.3.

Lemma 7.2. *For $p \in \mathscr{X}$, let $p_1, p_2 \in \mathscr{G}$ be two points such that $p \leftrightarrow p_1 \leftrightarrow p_2$ and $\textsc{Pair}(p_1), \textsc{Pair}(p_2) \in \mathrm{R}(\cdot\!:)_p$. Let ${}_p\square^{r_1}$ and ${}_p\square^{r_2}$ be the two unsatisfied rectangles that lead $\textsc{Greedy}$ to mark p_1 and p_2, respectively, while processing p (thus ${}^{r_1}\!\nwarrow\!\!\searrow_{r_2}$). Then there is a point $s \in \mathscr{X}$ such that ${}^{r_1}\!\nwarrow\!\!\searrow_{s}$ and ($\overset{s}{\underset{r_2}{\updownarrow}}$ or ${}^{s}\!\nwarrow\!\!\searrow_{r_2}$ or ${}_{r_2}\!\swarrow\!\!\nearrow^{s}$).*

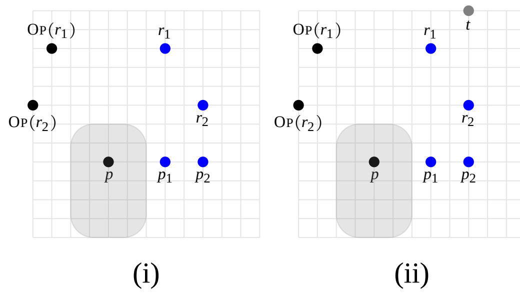


Figure 12: (i) The setting of Lemma 7.2: (1) $\textsc{Pair}(p_1), \textsc{Pair}(p_2) \in \mathrm{R}(\therefore)_p$ and (2) ${}_p\square^{r_1}$ and ${}_p\square^{r_2}$ are two unsatisfied rectangle encountered while processing p. (ii) Illustration of the case (4) in the proof when ${}_{r_1}\!\swarrow\!\!\nearrow^{\textsc{Up}(r_2)}$

.

Proof. Note that $r_1, r_2 \notin \mathscr{X}$ since $\textsc{Pair}(p_1), \textsc{Pair}(p_2) \in \therefore$. We consider four cases depending on the position of $\textsc{Up}(r_2)$:

1. $\textsc{Up}(r_2) = r_2$.

   Since $\textsc{Pair}(p_2) \in \therefore$, $r_2 \notin \mathscr{X}$. So $\textsc{Up}(r_2) \neq r_2$.

2. ${}^{r_1}\!\nwarrow\!\!\searrow_{\textsc{Up}(r_2)}$.

   In this case we can set $s := \textsc{Up}(r_2)$. So, we have found an original point s such that $({}^{r_1}\!\nwarrow\!\!\searrow_s$ and $\overset{s}{\underset{r_2}{\updownarrow}})$.

3. $r_1 \leftrightarrow \textsc{Up}(r_2)$.

   Since $\textsc{Pair}(p_1) \in \therefore$ and r_1 is the first point above p_1, hence $\textsc{Op}(r_1) \leftrightarrow r_1$. So $\textsc{Up}(r_2).y \neq r_1.y$; in particular $r_1 \leftrightarrow \textsc{Up}(r_2)$ cannot happen.

4. ${}_{r_1}\!\swarrow\!\!\nearrow^{\textsc{Up}(r_2)}$.

   Let t be the first point above r_2 such that ${}_{r_1}\!\swarrow\!\!\nearrow^{t}$ (see Fig. 12(ii)). Since we assumed that ${}_{r_1}\!\swarrow\!\!\nearrow^{\textsc{Up}(r_2)}$, $\textsc{Up}(r_2)$ is one such candidate. This implies that r_1 hides t from p(as $r_1 \in \ ({}_p\square^t)^\circ$). Let t' be the first point below t such that ${}^{r_1}\!\nwarrow\!\!\searrow_{t'}$. Such a point exists as r_2 is one such candidate. By Lemma 4.7, ${}^{r_1}\!\nwarrow\!\!\searrow_{\textsc{Op}(t')}$. In that case, $t' \neq r_2$ since ${}^{\textsc{Op}(r_2)}\!\nwarrow\!\!\searrow_p$ (since $\textsc{Pair}(p_2) \in \mathrm{R}(\therefore))$. So $\overset{t}{\underset{t'}{\updownarrow}}$ and $\overset{t'}{\underset{r_2}{\updownarrow}}$. This implies that either ${}^{\textsc{Op}(t')}\!\nwarrow\!\!\searrow_{r_2}$ or ${}_{r_2}\!\swarrow\!\!\nearrow^{\textsc{Op}(t')}$. So we have found an original point $s := \textsc{Op}(t')$ such that ${}^{r_1}\!\nwarrow\!\!\searrow_s$ and $({}^{s}\!\nwarrow\!\!\searrow_{r_2}$ or ${}_{r_2}\!\swarrow\!\!\nearrow^{s})$.

$\square$

The following lemma will allow us to use the k-decomposability of $\mathscr{X}$ to get at upper bound on $|\therefore|$.

Lemma 7.3. *Let $p \in \mathscr{X}$ and $\textsc{TopBlock}(p) = B$. Let $p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_\ell$ be such that $\textsc{Pair}(p_i) \in \mathrm{R}(\therefore)_p$ for all $1 \leqslant i \leqslant \ell$. Assume that* Greedy *marks points $p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_\ell$ due to unsatisfied rectangles ${}_p\square^{r_1}, {}_p\square^{r_2}, \ldots, {}_p\square^{r_\ell}$, respectively. By Lemma 6.10, $\textsc{Op}(r_i) \in B_i$ for $B_i \in \textsc{Sibling}(B)$. Then $B_i \neq B_j$ for $i \neq j$.*

Proof. Consider $p_i \leftrightarrow p_j$. One can check that $r_i.y < r_j.y$. By Lemma 7.2, there exists an original point s such that ${}^{r_i}\nwarrow\!\!\searrow_{s}$ and ($\overset{s}{\underset{r_j}{\updownarrow}}$ or ${}^{s}\nwarrow\!\!\searrow_{r_j}$ or ${}_{r_j}\swarrow\!\!\nearrow^{s}$).

Since ${}^{\textsc{Op}(r_i)}\nwarrow\!\!\searrow_{p}$ and ${}^{\textsc{Op}(r_j)}\nwarrow\!\!\searrow_{p}$ (as $\textsc{Pair}(p_i), \textsc{Pair}(p_j) \in \therefore$), this implies that ${}^{\textsc{Op}(r_i)}\nwarrow\!\!\searrow_{s}$ and ${}_{\textsc{Op}(r_j)}\swarrow\!\!\nearrow^{s}$. Also, we have $p \in B$ with $\textsc{Op}(r_i).x < p.x < s.x$ and $p \notin B_i$ (since B_i is a siblings of B). By the definition of a block, all the points in B_i should be contiguous on the x-axis. So, $s \notin B_i$.

Thus we have $s \notin B_i$ and $\textsc{Op}(r_i).y < s.y < \textsc{Op}(r_j).y$. Again, by the definition of a block, all the points in B_i should have contiguous time interval. So, $\textsc{Op}(r_j)$ does not lie in B_i and $B_i \neq B_j$.

$\square$

Since there are at most $k-1$ sibling of B, by Lemma 7.3, $|\mathrm{R}(\therefore)_p| \leqslant k-1$. So, $|\mathrm{R}(\therefore)| = \sum_{p \in \mathscr{X}} |\mathrm{R}(\therefore)_p| \leqslant n(k-1)$. Similarly, we can show that $|\mathrm{L}(\therefore)| \leqslant n(k-1)$. Since $|\therefore| = |\mathrm{R}(\therefore)| + |\mathrm{L}(\therefore)|$, we have proved Theorem 7.1.

8 Improving the bound on $|\therefore|$

In this section we prove:

Theorem 8.1. $|\therefore| \leqslant 14n\log k$.

To improve the the bound on $|\therefore|$, we need to show some more properties of a block in $\textsc{TreeDecomposition}(\mathscr{X})$.

8.1 Properties of a Block

Let $\textsc{MaxTime}(B) := argmax_{z \in B}\{z.y\}$. That is, $\textsc{MaxTime}(B)$ is an original point in B that has the maximum y-coordinate. We can similarly define $\textsc{MinTime}(B)$. Note that $\textsc{Top}(B) = \textsc{MinTime}(B)$. Let $\textsc{MaxKey}(B) := argmax_{z \in B}\{z.x\}$ and $\textsc{MinKey}(B) := argmin_{z \in B}\{z.x\}$.

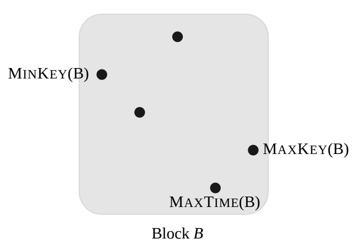


Figure 13: Pictorial representation of $\textsc{MaxTime}(B), \textsc{MaxKey}(B)$ and $\textsc{MinKey}(B)$

Definition 8.2. *(Left and Right points of block B) Let* $\textsc{TopBlock}(p) = B$. *Define* $\textsc{Left}(B)$ *to be the first point to the left of p marked by* $\textsc{Greedy}$ *while processing p. Define* $\textsc{Right}(B)$ *in a symmetric fashion.*

Note that $\textsc{Left}(B)$, if it exists, satisfies $\textsc{Left}(B).x < \textsc{MinKey}(B).x$ since by Observation 4.1, $\textsc{Greedy}$ cannot put any point above any other original point in B. Similarly $\textsc{Right}(B).x > \textsc{MaxKey}(B).x$. This implies that $\textsc{Greedy}$ does not put any point in $\textsc{Box}(B)$ while processing $\textsc{Top}(B)$.

Observation 8.3. $\textsc{Greedy}$ *does not put any marked points in* $\textsc{Box}(B)$ *while processing* $\textsc{Top}(B)$.

Lemma 8.4. *Let* $\textsc{TopBlock}(r) = B'$ *and* $\textsc{Parent}(B') = B$. *So* $r \neq \textsc{Top}(B)$. *While processing* r, $\textsc{Greedy}$ *can put marked point (1)Below* $\textsc{Left}(B)$ *or/and (2)Below* $\textsc{Right}(B)$ *or/and (3) In* $\textsc{Box}(B)$.

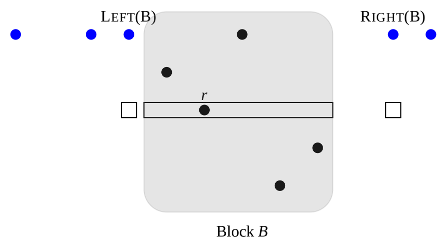


Figure 14: $\textsc{Left}(B)$ and $\textsc{Right}(B)$ of a block B. Note that $\textsc{Greedy}$ cannot put any point in $\textsc{Box}(B)$ while processing $\textsc{Top}(B)$. Lemma 8.4 shows that while processing any point $r \neq \textsc{Top}(B)$, $\textsc{Greedy}$ can put points only in the rectangles shown in the figure.

Proof. Assume that $\textsc{Greedy}$ puts a point r_1 while processing r. Consider the following two cases for contradiction:

1. $r_1.x < \textsc{Left}(B).x$

   $\textsc{Up}(r_1)$ cannot satisfy the property ${}_{\textsc{Up}(r_1)}\swarrow\!\!\!\nearrow^{\textsc{Left}(B)}$ as then $\textsc{Up}(r_1).x < \textsc{MinKey}(B).x$. By definition of decomposability, all the original points with time between $[\textsc{Top}(B).y, \textsc{MaxTime}(B).y]$ should have their key value $\geqslant \textsc{MinKey}(B).x$. However, $\textsc{Up}(r_1).x < \textsc{Left}(B).x < \textsc{MinKey}(B).x$.

   So assume that ${}^{\textsc{Up}(r_1)}\nwarrow\!\!\!\searrow_{\textsc{Left}(B)}$. Let r' be the first point above r_1 with the property that $({}^{r'}\nwarrow\!\!\!\searrow_{\textsc{Left}(B)}$ or $r' \leftrightarrow \textsc{Left}(B))$. Such a point exists as $\textsc{Up}(r_1)$ is one such candidate. This implies that $\textsc{Left}(B)$ hides r' from r as $\textsc{Left}(B)$ lies in ${}^{r'}\square_{r}$. Let r'' be the first point below r'. Such a point exists as r_1 itself is one such candidate. Also, note that due to the way r' is defined, ${}_{r''}\swarrow\!\!\!\nearrow^{\textsc{Left}(B)}$. By Lemma 4.6, ${}_{\textsc{Up}(r'')}\swarrow\!\!\!\nearrow^{\textsc{Left}(B)}$. This again leads to contradiction as $\textsc{Up}(r'').y$ is between $[\textsc{Top}(B).y, \textsc{MaxTime}(B).y]$ and its key value $\textsc{Up}(r'').x < \textsc{Left}(B).x < \textsc{MinKey}(B).x$.

2. $\textsc{Left}(B).x < r_1.x < \textsc{MinKey}(B).x$

   Let r' be a point above r_1 with least y co-ordinate and the property that ${}^{\textsc{Left}(B)}\nwarrow\!\!\!\searrow_{r'}$. If no such point exists then define $r' := r_1$. Let $\textsc{Greedy}$ mark point r' while processing $\textsc{Op}(r') \in B$ due to an unsatisfied rectangle ${}^{r''}\square_{\textsc{Op}(r')}$. By definition of $\textsc{Left}(B)$, r'' cannot lie to the right of $\textsc{Left}(B)$. So, ${}_{\textsc{Left}(B)}\swarrow\!\!\!\nearrow^{r''}$. This implies that there exists a non-top element of B, $\textsc{Op}(r')$ with $\textsc{Pair}(r') = (\textsc{Op}(r'), \textsc{Op}(r''))$ and $\textsc{Op}(r'') \notin B$. This contradict Lemma 6.9

The case when $r_1.x > \textsc{Right}(B).x$ and $\textsc{MaxKey}(B).x > r_1.x > \textsc{Right}(B).x$ are symmetric to above two cases respectively. $\square$

We now need to calculate the number of points added by $\textsc{Greedy}$ in $\textsc{Box}(B)$ while processing r. To this end, we define some notations.

Definition 8.5. *Let* $\textsc{Reg}(B) := \{(a,b) | \textsc{Up}(a) \in B$ *and* $b > \textsc{MaxTime}(B).y$ *and* $b < \textsc{MaxTime}(\textsc{Parent}(B)).y\}$

In words, $\textsc{Reg}(B)$ denote the region below $\textsc{Box}(B)$ till the last original point in $\textsc{Parent}(B)$. If $\mathscr{B}$ is the set of blocks, then define $\textsc{Reg}(\mathscr{B}) := \cup_{B \in \mathscr{B}} \textsc{Reg}(B)$.

Lemma 8.6. *Let* $\textsc{TopBlock}(r) = B'$ *and* $\textsc{Parent}(B') = B$. *Then while processing* r, $\textsc{Greedy}$ *can put marked point only (1)below* $\textsc{Left}(B)$ *and/or (2)Below* $\textsc{Right}(B)$ *and/or (3) in* $\textsc{Reg}(B'')$ *where* $B'' \in \textsc{Sibling}(B')$.

Proof. By Lemma 8.4, while processing r, $\textsc{Greedy}$ can put marked point in $\textsc{Box}(B)$, below $\textsc{Left}(B)$ and below $\textsc{Right}(B)$). However by observation 8.3, $\textsc{Greedy}$ cannot put any marked point in $\textsc{Box}(B')$ while processing r.

Let $B'' \in \textsc{Sibling}(B')$. By Lemma 5.1, $\textsc{Greedy}$ cannot put any marked point in $\textsc{UpperBox}(B'')$ while processing r. Also, $\textsc{Greedy}$ cannot put any marked point in $\textsc{Box}(B'')$ while processing r. This implies that while processing r, all the points marked by $\textsc{Greedy}$ in $\textsc{Box}(B)$ are in $\textsc{Reg}(B'')$ where $B'' \in \textsc{Sibling}(B')$. ☐

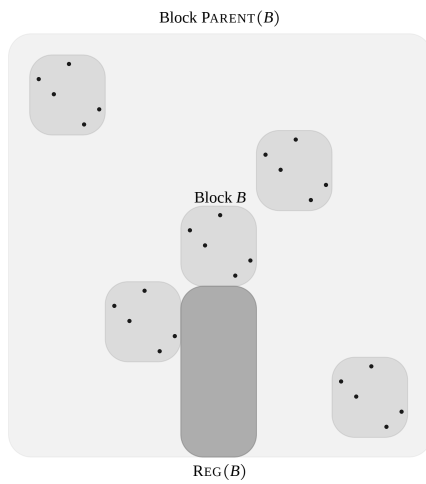


Figure 15: B has four siblings. $\textsc{Reg}(B)$ is the region shaded below $\textsc{Box}(B)$

We will now describe properties of the points that are added in $\textsc{Reg}(B)$.

Definition 8.7. *A point* p *is a key-new point in* $\textsc{Reg}(B)$ *if there is no point* $q \in \textsc{Reg}(B)$ *such that (1)*$p.x = q.x$ *and* $q.y < p.y$, *else it is called key-old.*

In other words, if no point exists in $\textsc{Reg}(B)$ with the same key as $p.x$ before time $p.y$, then p is key-new.

Definition 8.8. *Let* $p \in \mathscr{X}$ *be an original point with least* $p.y$ *and the property (1)* $p.y > \textsc{MaxTime}(B).y$ *and (2)*$p.x < \textsc{Left}(B)$ *or* $\textsc{Left}(B) < p.x < \textsc{MinKey}(B)$. *Then we say that* p *is the left-relative of* B *or* $p \in \textsc{Left-Rel}(B)$. *Similarly define* $\textsc{Right-Rel}(B)$ *in a symmetric fashion.*

In words, $\textsc{Left-Rel}(B)$ contains the first original point in the sequence $\mathscr{X}$ that comes after all the points in B and has key strictly than $\textsc{Left}(B).x$ or in the range $[\textsc{Left}(B).x, \textsc{MinKey}(B).x]$. Note that there are at most 2 original points in the set $\textsc{Left-Rel}(B)$.

Lemma 8.9. *Let q be an original point that satisfies $q.x < \textsc{Left}(B)$ and $q.y > p.y$ where $p \in \textsc{Left-Rel}(B)$ and $p.x < \textsc{Left}(B).x$. Then* GREEDY *cannot mark any key-new point in* $\textsc{Reg}(B)$ *while processing q.*

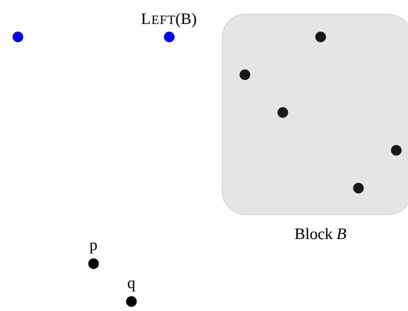


Figure 16: $p \in \textsc{Left-Rel}(B)$. Lemma 8.9 states that while processing any other point q that arrives after p and is to the left of $\textsc{Left}(B)$, GREEDY cannot mark any key-new point in $\textsc{Reg}(B)$.

Proof. Assume for contradiction that GREEDY put a key-new point s while processing q in $\textsc{Reg}(B)$ due to unsatisfied rectangle ${}_q\square^{s'}$. This implies that $\textsc{Pair}(s) = (q, \textsc{Op}(s'))$. Since we have assumed that s is a key-new point s' cannot lie in $\textsc{Reg}(B)$, so it lies above $\textsc{Reg}(B)$. By Lemma 5.1, no point can lie above $\textsc{Box}(B)$. So s' lies in $\textsc{Box}(B)$. We first note that $\textsc{Op}(s') \neq \textsc{Top}(B)$ as then ${}_q\square^{s'}$ is satisfied by $\textsc{Left}(B)$.

Let us assume that $\textsc{Op}(s') \neq \textsc{Top}(B)$. Since ${}_q\square^{s'}$ was an unsatisfied rectangle, there exists no point in it when GREEDY processes q. Consider the point $p \in \textsc{Left-Rel}(B)$ with $p.x < \textsc{Left}(B).x$. By definition, $p.y < q.y$. Consider the rectangle ${}_p\square^{s'}$. We claim that ${}_p\square^{s'}$ is an unsatisfied rectangle as (1) there are no point in ${}_q\square^{s'}$ when GREEDY processes p and (2) By Lemma 8.4, for any point $r \neq \textsc{Top}(B)$, GREEDY does not put any point to the left of $\textsc{Left}(B)$ and (3) There is no point r with the property that $r.y > \textsc{MaxTime}(B).y$ and ${}_p\swarrow\!\!\!\nearrow^r$ and ${}^r\nwarrow\!\!\!\searrow_q$. Indeed, if such a point exists, then define r be a point which satisfy the above property and has the least y-coordinate. If ${}^{\textsc{Op}(r)}\nwarrow\!\!\!\searrow_q$, then $\textsc{Op}(r) \in \textsc{Left-Rel}(b)$ and not p which contradicts our assumption. $\textsc{Op}(r)$ cannot lie in ${}_q\square^{s'}$ and in $\textsc{Reg}(B)$. So ${}^{\textsc{MaxTime}(B)}\nwarrow\!\!\!\searrow_{\textsc{Op}(r)}$ and GREEDY markes the point r while processing $\textsc{Op}(r)$ due to unsatisfied rectangle, say ${}^{r'}\square_{\textsc{Op}(r)}$. Note that ${}_{r'}\swarrow\!\!\!\nearrow^{\textsc{MaxTime}(B)}$, as otherwise ${}^{r'}\square_{\textsc{Op}(r)}$ is satisfied by $\textsc{MaxTime}(B)$. However, this implies that r' is the point with least y-coordinate and the property that $r'.y > \textsc{MaxTime}(B).y$ ${}_p\swarrow\!\!\!\nearrow^{r'}$ and ${}^{r'}\nwarrow\!\!\!\searrow_q$. This contradicts the definition of r.

This implies that ${}_p\square^{s'}$ is unsatisfied when GREEDY processes p. So GREEDY should put a point, say s'' below s'. However, this implies that s'' is a key-new point and not s contradicting the assumption of the lemma. ☐

Similar to the above lemma one can also prove the following lemma.

Lemma 8.10. *Let q be an original point that satisfies $\textsc{Left}(B) < q.x < \textsc{MinKey}(B)$ and $q.y > p.y$ where $p \in \textsc{Left-Rel}(B)$ and $\textsc{Left}(B) < p.x < \textsc{MinKey}(B)$. Then* GREEDY *cannot mark any key-new point in* $\textsc{Reg}(B)$ *while processing q.*

Similarly one can prove the symmetric versions of the above two lemmas. We now describe the implications of the above two lemmas. The lemma suggest that at most four points (2 in $\textsc{Left-Rel}(B)$ and 2 in $\textsc{Right-Rel}(B)$) can be responsible for adding key-new points in $\textsc{Reg}(B)$.

Corollary 8.11. GREEDY *can mark key-new points in* $\text{REG}(B)$ *while processing points in* $\text{LEFT-REL}(B) \cup \text{RIGHT-REL}(B)$ *only.*

We define some more properties of the points added to $\text{REG}(B)$.

Definition 8.12. *A key b is live in $\text{REG}(B)$ at time t if there is a point $p \in \text{REG}(B)$ such that $p.x = b$ and $p.y < t$ and there exists no point $p' \in \text{REG}(B)$ such that (1) $p'.x < p.x$ or $p'.x > p.x$ and (2) $t > p'.y \geqslant p.y$. We say that key b is not-live in $\text{REG}(B)$ if such a point p' exists. A point p is said to be key-live at time t in $\text{REG}(B)$ if key $p.x$ is live in $\text{REG}(B)$ at time t, other-wise it is key-not-live.*

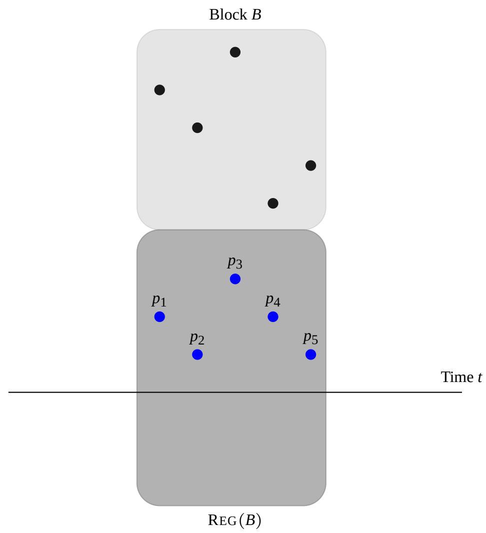


Figure 17: Key $p_1.x, p_2.x$ and $p_5.x$ are live at time t and Key $p_3.x$ and $p_4.x$ are not live. Similarly point p_1, p_2 and p_5 are key-live and point p_3 and p_4 are key-not-live.

By Lemma 8.9, only points in $\text{LEFT-REL}(B) \cup \text{RIGHT-REL}(B)$ can add key-new points to $\text{REG}(B)$. Let $p \in \text{LEFT-REL}(B)$ such that GREEDY adds a key-new point q in $\text{REG}(B)$ while processing p due to unsatisfied rectangle $_p\square^{q'}$. Since q is key-new in $\text{REG}(B)$, q' must lie in $\text{BOX}(B)$. This implies that $\text{OP}(q') \in B$. Since p is a left-relative of B, it lies to the left of $\text{MINKEY}(B)$. This implies that $\text{PAIR}(q) \in$ ⁖. So, we have proved the following lemma:

Lemma 8.13. *All key-new points added by original points in* $\text{LEFT-REL}(B)$ *in* $\text{REG}(B)$ *have* $\text{PAIR}(\cdot) \in$ ⁖.

One can also prove the symmetric version of the above lemma in a similar way.

Lemma 8.14. *All key-new points added by original points in* $\text{RIGHT-REL}(B)$ *in* $\text{REG}(B)$ *have* $\text{PAIR}(\cdot) \in$ ⁖.

8.2 Improving the bound on |⁘| to $O(n \log k)$

Let B be any block of $\text{TREEDECOMPOSITION}(\mathscr{X})$ that has at most $\ell (\ell < k)$ children. Let these children be $B_1, B_2, \ldots, B_\ell$ where $\text{MAXKEY}(B_i).x < \text{MINKEY}(B_{i+1}).x$, or in words, each key of the block

B_i is smaller than each key of block B_{i+1}. We recursively partition ℓ blocks into two half till a singleton block is obtained. This partition can be represented as a tree called the $\textsc{Partition}(B)$. At the root of $\textsc{Partition}(B)$ is the region $\textsc{Reg}(B_1, B_2, \ldots, B_\ell)$. We then divide this region into two half[2] : $\textsc{Reg}(B_1, B_2, \ldots, B_{\ell/2})$ and $\textsc{Reg}(B_{\ell/2+1}, B_{\ell/2+2}, \ldots, B_\ell)$. In Lemma 8.15, we will show that the total number of points (with $\textsc{Pair}(\cdot) \in \therefore$) added by $\textsc{Greedy}$ in $\textsc{Reg}(B_{\ell/2+1}, B_{\ell/2+2}, \ldots, B_\ell)$ while processing points in $\{\textsc{Top}(B_1), \textsc{Top}(B_2), \ldots, \textsc{Top}(B_{\ell/2})\} \backslash \textsc{Top}(B)$ is $O(\ell)$. Similarly, the total number of points (with $\textsc{Pair}(\cdot) \in \therefore$) added by $\textsc{Greedy}$ in $\textsc{Reg}(B_1, B_2, \ldots, B_\ell)$ while processing original points in $\{\textsc{Top}(B_{\ell/2+1}), \textsc{Top}(B_{\ell/2+2}), \ldots, \textsc{Top}(B_\ell)\} \backslash \textsc{Top}(B)$ is $O(\ell)$. Note that the top element of the block B is exempted as it cannot add any point in the $\textsc{Reg}(B_1, B_2, \ldots, B_\ell)$. In general we have to show the following lemma:

Lemma 8.15. *Let $B_i, B_{i+1}, \ldots, B_{i+2m-1}$ be the set of consecutive children blocks of B in $\textsc{TreeDecomposition}(\mathscr{X})$. The number of points added by $\{\textsc{Top}(B_i), \textsc{Top}(B_{i+1}), \ldots, \textsc{Top}(B_{i+m-1})\} \backslash \textsc{Top}(B)$ in $\textsc{Reg}(B_{i+m}, B_{i+m+1}, \ldots, B_{i+2m-1})$ with $\textsc{Pair}(\cdot) \in \therefore$ is $\leqslant 12m$.*

By symmetry one can also show that the number of points added by $\{\textsc{Top}(B_{i+m}), \textsc{Top}(B_{i+m+1}), \ldots, \textsc{Top}(B_{i+2m-1})\} \backslash \textsc{Top}(B)$ in $\textsc{Reg}(B_i, B_{i+1}, \ldots, B_{i+m-1})$ with $\textsc{Pair}(\cdot) \in \therefore$ is $\leqslant 12m$.

Let $Y(B) := T(B_1, B_2, \ldots, B_\ell)$ be the total number of points (with $\textsc{Pair}(\cdot) \in \therefore$) added by $\textsc{Greedy}$ while processing points in $\{\textsc{Top}(B_1), \textsc{Top}(B_2), \ldots, \textsc{Top}(B_\ell)\} \backslash \textsc{Top}(B)$ in $\textsc{Reg}(B_1, B_2, \ldots, B_\ell)$. It can be calculated as follows: $Y(B) = T(B_1, B_2, \ldots, B_\ell) = T(B_1, B_2, \ldots, B_{\ell/2}) + T(B_{\ell/2+1}, B_{\ell/2+2}, \ldots, B_\ell) + 12\ell$. This would imply that $Y(B) = T(B_1, B_2, \ldots, B_\ell) \leqslant 12\ell \log \ell$.

We would charge these $12\ell \log \ell$ points to the following $\ell - 1$ original points: $\{\textsc{Top}(B_1), \textsc{Top}(B_2), \ldots, \textsc{Top}(B_\ell)\} \backslash \textsc{Top}(B)$. That is, each top point of children of block B except one gets $\leqslant 13 \log \ell$ charge.

We are now ready to prove Theorem 8.1

Proof. Let $\textsc{TopBlock}(r) = B'$ and $\textsc{Parent}(B') = B$. By Lemma 8.4, $\textsc{Greedy}$ can put at most two point below $\textsc{Left}(B)$ and $\textsc{Right}(B)$. And by the analysis above, the amortized number of points (with $\textsc{Pair}(\cdot) \in \therefore$) added by $\textsc{Greedy}$ in $\textsc{Box}(B)$ while processing r with $\textsc{Pair}(\cdot) \in \therefore$ is $13 \log \ell \leqslant 13 \log k$ where ℓ is the number of children of B. So, amortized number of points added by $r = 2 + 13 \log k$. So $|\therefore| \leqslant 14 n \log k$. $\square$

The rest of the section is devoted in proving Lemma 8.15.

8.3 Proof of Lemma 8.15

Let $B_1, B_2, \ldots, B_l$ be the children of B in $\textsc{TreeDecomposition}(\mathscr{X})$. Let $B_i, B_{i+1}, \ldots, B_{i+2m-1}$ be the consecutive $2m$ children of B. Let $\mathscr{B}_\ell = \{B_i, B_{i+1}, \ldots, B_{i+m-1}\}$ and $\mathscr{B}_r = \{B_{i+m}, B_{i+m+1}, \ldots, B_{i+2m-1}\}$. Let $\mathscr{B}_{ll} = \{B_1, B_2, \ldots, B_{i-1}\}$ and $\mathscr{B}_{rr} = \{B_{i+2m}, B_{i+2m+1}, \ldots, B_\ell\}$. Let C_t denote the set of key-live points in $\textsc{Reg}(\mathscr{B}_r)$ after $\textsc{Greedy}$ finished processing all the points till step t.

Lemma 8.16. *Let $B_j \in \{\mathscr{B}_\ell, \mathscr{B}_{\ell\ell}\}$. Let $N_{\textsc{Top}(B_j)}, R_{\textsc{Top}(B_j)}$ be the set of key-new and key-old points added by $\textsc{Greedy}$ in $\textsc{Reg}(\mathscr{B}_r)$ while processing $\textsc{Top}(B_j)$. Then $C_{\textsc{MaxTime}(B_j).y+1} \leqslant C_{\textsc{MinTime}(B_j).y} - \max\{0, (|R_{\textsc{Top}(B)}| - 2)\} + 2|\{B_k \in \mathscr{B}_r | \textsc{Top}(B) \in \textsc{Left-Rel}(B_k)\}|$.*

[2] we drop the floor notation for readability

Proof. Let $p_1 \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_n$ be the set of points added by GREEDY in $\textsc{Reg}(\mathscr{B}_r)$ while processing $\textsc{Top}(B_j)$. Note that the key $\{p_1.x, p_2.x, \ldots, p_n.x\}$ are live in $\textsc{Reg}(\mathscr{B}_r)$ at time $\textsc{Top}(B_j).y$. By definition, only p_1 and p_n can be live at time $\textsc{Top}(B_j).y+1$ as each other point p_i is hidden by the point p_{i-1} and p_{i+1}. p_1 and p_n can be key-new or key-old point. Consider only the points in $R_{\textsc{Top}(B_j)}$. All the keys in $R_{\textsc{Top}(B_j)}$ are live at time $\textsc{Top}(B_j).y$ and only 2 of these (p_1 and p_n) can be live at time $\textsc{Top}(B_j).y+1$. So, the number of key-live points decrease at least by $\max\{0, |R_{\textsc{Top}(B_j)}|-2\}$ after processing $\textsc{Top}(B_j)$.

Consider the points in $N_{\textsc{Top}(B_j)}$. By Lemma 8.9, if $\textsc{Top}(B_j) \in \textsc{Left-Rel}(B_k)$ for $B_k \in \mathscr{B}_r$, then GREEDY can add key-new points out of which only only at most two (possibly p_1 and p_n) are live. So the total number of key-live points added by GREEDY while processing p is at most $2|\{B_k \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Left-Rel}(B_k)\}|$. This implies $C_{\textsc{Top}(B_j).y} \leqslant C_{\textsc{Top}(B_j).y-1} - \max\{0, (|R_{\textsc{Top}(B_j)}|-2)\} + 2|\{B_k \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Left-Rel}(B_k)\}|$.

Consider a point $q \in B_j$ such that $q \neq \textsc{Top}(B_j)$. By Lemma 8.4, all the points added by GREEDY while processing q are either in $\textsc{Box}(B_j)$ or below $\textsc{Left}(B_j)$ and $\textsc{Right}(B_j)$. Note that only $\textsc{Right}(B_j)$ can lie in $\textsc{Reg}(\mathscr{B}_r)$. Whenever GREEDY adds a point below $\textsc{Right}(B_j)$, key $\textsc{Right}(B_j).x$ is live at time $q.y$. So GREEDY does not increase or decrease key-live points while processing q. This implies that $C_{\textsc{MaxTime}(B_j).y+1} \leqslant C_{\textsc{Top}(B_j).y} - \max\{0, (|R_{\textsc{Top}(B_j)}|-2)\} + 2|\{B_j \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Left-Rel}(B_j)\}|$ ☐

Similar, we can prove the following lemma.

Lemma 8.17. *Let $B_j \in \{\mathscr{B}_{rr}\}$. Let $N_{\textsc{Top}(B_j)}, R_{\textsc{Top}(B_j)}$ be the set of key-new and key-old points added by* GREEDY *in* $\textsc{Reg}(\mathscr{B}_r)$ *while processing* $\textsc{Top}(B_j)$. *Then* $C_{\textsc{MaxTime}(B_j).y+1} \leqslant C_{\textsc{Top}(B_j).y} - \max\{0, (|R_{\textsc{Top}(B_j)}|-2)\} + 2|\{B_j \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Right-Rel}(B_j)\}|$.

Lemma 8.18. *Let $B_j \in \mathscr{B}_r$. Then* $C_{\textsc{MaxTime}(B_j).y+1} \leqslant C_{\textsc{MinTime}(B_j).y} + 4$.

Proof. Let $p = \textsc{Top}(B_j)$. Let $p_1 \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_n$ be the set of points added by GREEDY in $\textsc{Reg}(\mathscr{B}_r)$ while processing p. By definition, only p_1 and p_n can shift their status from key-not-live to key-live at time $p.y+1$ as each other point p_i is hidden by the point p_{i-1} and p_{i+1}. Let $q \in B_j$ such that $q \neq \textsc{Top}(B_j)$. By Lemma 8.4, all the points added by GREEDY while processing q are either in $\textsc{Box}(B_j)$ or below $\textsc{Left}(B_j)$ and $\textsc{Right}(B_j)$. Note that all the points that are added in $\textsc{Box}(B_j)$ are neither key-live or key-not-live as they do not lie in $\textsc{Reg}(\mathscr{B}_r)$. So, they cannot change their status from key-not-live to key-live. So after processing point $\textsc{MaxTime}(B_j)$, at most 4 keys $p_1.x, p_n.x, \textsc{Left}(B_j).x, \textsc{Right}(B_j).x$ can shift their status from key-live to key-not-live. This implies that $C_{\textsc{MaxTime}(B_j).y+1} \leqslant C_{\textsc{MinTime}(B_j).y} + 4$. ☐

By Lemma 8.16, if $B_j \in \mathscr{B}_\ell$, $C_{\textsc{MaxTime}(B_j).y+1} \leqslant C_{\textsc{Top}(B_j).y} - (|R_{\textsc{Top}(B_j)}|-2) + 2|\{B_k \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Left-Rel}(B_k)\}|$. By Lemma 8.16, if $B_j \in \mathscr{B}_{\ell\ell}$, $C_{\textsc{MaxTime}(B_j).y+1} \leqslant C_{\textsc{Top}(B_j).y} + 2|\{B_k \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Left-Rel}(B_k)\}|$. By Lemma 8.17, if $B_j \in \mathscr{B}_{rr}$, $C_{\textsc{MaxTime}(B_j).y+1} \leqslant C_{\textsc{Top}(B_j).y} + 2|\{B_k \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Right-Rel}(B_k)\}|$

The above two sums along with the inequality in Lemma 8.18 are telescoping sums that starts at the top of the parent block B and ends at $\textsc{MaxTime}(B)$. Adding them gives the following expression.

$C_{\textsc{MaxTime}(B).y+1} - C_{\textsc{Top}(B).y} \leqslant -\sum_{B_j \in \mathscr{B}_\ell}(|R_{\textsc{Top}(B_j)|}-2) + \sum_{B_j \in \{\mathscr{B}_\ell, \mathscr{B}_{\ell\ell}\}} 2|\{B_k \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Left-Rel}(B_k)\}| + \sum_{B_j \in \mathscr{B}_{rr}} 2|\{B_k \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Right-Rel}(B_k)\}| + \sum_{B_j \in \mathscr{B}_r} 4$. Since, there are no points in $\textsc{Reg}(\mathscr{B}_r)$ at time $\textsc{Top}(B).y$, $C_{\textsc{Top}(B).y} = 0$. Also there can at most be two points in $\textsc{Left-Rel}(B_k)$ and $\textsc{Right-Rel}(B_k)$, so $2|\{B_k \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Left-Rel}(B_k)\}| + 2|\{B_k \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Right-Rel}(B_k)\}| \leqslant 4\sum_{B_k \in \mathscr{B}_r} 2 = 8m$. This leads to the following inequality: $0 \leqslant -\sum_{B_j \in B_\ell}(|R_{\textsc{Top}(B_j)}|) + 8m + 4m$. So, $\sum_{B_j \in B_\ell}(|R_{\textsc{Top}(B_j)}|) \leqslant 12m$. While processing $\textsc{Top}(B_j)$, all the points added by GREEDY other than $R_{\textsc{Top}(B_j)}$ are key-new. By Lemma 8.13, these points are in . Note that even points in $R_{\textsc{Top}(B_j)}$ can be in . However, we have shown that number of such points is $\leqslant 12m$. Thus we have proved Lemma 8.15.

9 Upper Bounding $|.\!:\!\cdot|$: Good Pairs

To prove that GREEDY has small competitive ratio we need to show that $|\mathscr{G}|/|\mathscr{X} \cup \mathrm{OPT}(\mathscr{X})|$ is small. But our understanding of this ratio remains quite limited with the best upper bound being $O(\log n)$. One way to prove that GREEDY has small competitive ratio, sidestepping the above issue, would be to prove a *lower* bound on $\mathrm{OPT}(\mathscr{X})$ (the minimum cardinality point set that must be added to $\mathscr{X}$ to make $(\mathscr{X} \cup \mathrm{OPT}(\mathscr{X}))$ arborally satisfied) by constructing a *certificate* of the lower bound and then show that the the ratio between $|\mathscr{G}|$ and the lower bound is small. While working with the lower bound might give a worse guarantee, but it might also allow more flexibility in the proof. This is a standard approach in proving approximation ratios of algorithms. For the BST problem, there are many lower bounds known (see [5]). One lower bound, called the independent set lower bound from Demaine et al. [5] subsumes the previous ones. It is defined as follows: rectangles (p,q) and (r,s) with $p,q,r,s \in \mathscr{X}$, are *independent* (in $\mathscr{X}$) if they are not arborally satisfied and no corner of either rectangle is strictly inside the other. They showed that the cardinality of any independent set of rectangles provides a lower bound on $|\mathscr{X} \cup \mathrm{OPT}(\mathscr{X})|$ as follows:

Claim 9.1 (Claim 4.1, [5])**.** *Let $\mathscr{X}$ contain an independent set of rectangles I and let $\mathrm{OPT}(\mathscr{X})$ be a minimum cardinality point set that must be added to $\mathscr{X}$ to make $(\mathscr{X} \cup \mathrm{OPT}(\mathscr{X}))$ arborally satisfied, then $|\mathscr{X} \cup \mathrm{OPT}(\mathscr{X})| \geqslant |I|/2 + |\mathscr{X}|$.*

Though the above lemma gives a lower bound, it is not clear how to construct the set I (lower bound certificate), or to relate it to $\mathscr{G}$. Demaine et al. provided an alternative lower bound that is efficiently computable by a procedure called SIGNEDGREEDY which is very similar to GREEDY and is within a constant factor of the best independent set lower bound. However, it is not known how to relate this lower bound to $|\mathscr{G}|$, or how to relate the executions of SIGNEDGREEDY and GREEDY despite their close similarity.

Our work provides a lower bound that can be related to $|\mathscr{G}|$ on k-decomposable sequences. The lower bound certificate is constructed by directly looking at the execution of GREEDY. Our construction builds upon the idea of the independent set lower bound [5], but the final construction does not provide an independent set, but a more nuanced certificate. The above features of our technique give us hope that our techniques can be refined to better understand the performance of GREEDY.

In Sec. 6, Observation 6.4 associated each pair in $.\!:\!\cdot$ with a rectangle. The set of rectangles thus formed (associated with the pairs in $.\!:\!\cdot$) is tightly coupled with the execution of GREEDY. We will first partition $.\!:\!\cdot$ into two parts: (1) $\mathrm{GOOD}(.\!:\!\cdot)$ and (2) $\mathrm{BAD}(.\!:\!\cdot)$. While the set $\mathrm{GOOD}(.\!:\!\cdot)$ can be quite different from independent sets, we show that $\mathrm{GOOD}(.\!:\!\cdot)$ behaves like an independent set in the following sense:

Theorem 9.2. *Let $\mathscr{X}$ be the original point set and let $\mathrm{OPT}(\mathscr{X})$ be the minimum number of points that must be added to $\mathscr{X}$ to make it arborally satisfied. Then $|\mathscr{X} \cup \mathrm{OPT}(\mathscr{X})| \geqslant \frac{|\mathrm{GOOD}(.\!:\!\cdot)|}{2} + |\mathscr{X}|$.*

The rest of this section is devoted in proving Theorem 9.2.

9.1 Good Pairs and their properties

We will now partition $.\!:\!\cdot$ into two parts:

Definition 9.3. *(Good and bad pairs)* $\mathrm{PAIR}(p_1) = (p,q)$ *with* $\mathrm{PAIR}(p_1) \in .\!:\!\cdot$ *is called a* good *pair if* $r \notin ({}^{q}\square_{p})^{\circ}$ *(or* $({}_{p}\square^{q})^{\circ}$*) for all* $r \in \mathscr{X}$*. Otherwise,* $\mathrm{PAIR}(p_1) = (p,q)$ *is called a* bad *pair. Let* $\mathrm{GOOD}(.\!:\!\cdot) := |\{p \in \mathscr{G} \mid \mathrm{PAIR}(p)$ *is a good pair*$\}|$ *and* $\mathrm{BAD}(.\!:\!\cdot) := |\{p \in \mathscr{G} \mid \mathrm{PAIR}(p)$ *is a bad pair*$\}|$.

Towards the end of proving Theorem 9.2, we will show some properties of good pairs in the next lemma. The definition of a good pair forbids any point from $\mathscr{X}$ being in the interior of the rectangle. The following lemma proves that points from $\mathscr{G}$ also cannot lie in the interior of the rectangle.

Lemma 9.4. *Let* $\textsc{Pair}(p_1) = {}^{p}\square_{q}$ *(or* ${}_{p}\square^{q}$*) be a good pair. Then there are no points in* $({}^{p}\square_{q})^{\circ}$ *(or* $({}_{p}\square^{q})^{\circ}$*).*

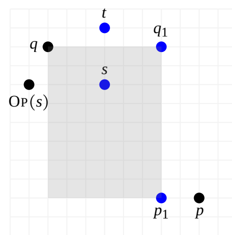


Figure 18: Illustration of the case dealt in the proof of Lemma 9.4 when s lies in ${}^{q}\square_{p_1}$.

Proof. By symmetry, it suffices to prove the lemma for rectangles of type ${}^{q}\square_{p}$. Note that no original point can be in $({}^{q}\square_{p})^{\circ}$ as $\textsc{Pair}(p_1)$ is a good pair. Let q_1 be the first point above p_1. This implies that $\textsc{Greedy}$ marks point p_1 to the left of p due to the unsatisfied rectangle ${}^{q_1}\square_{p}$.

If $q_1 = \textsc{Op}(q_1)(= q)$, then there are no points in $({}^{q}\square_{p})^{\circ}$ as $\textsc{Greedy}$ found ${}^{q_1}\square_{p}$ arborally unsatisfied. So let us assume that $q \leftrightarrow q_1$.

If there is a point $s \in ({}^{q}\square_{p})^{\circ}$ then it must satisfy one of the following three cases (recall that we must have $s \in \mathscr{G}$):

1. $s \in ({}^{q_1}\square_{p})^{\circ}$.

   Again, there exists no point in $({}^{q_1}\square_{p})^{\circ}$ as $\textsc{Greedy}$ found ${}^{q_1}\square_{p}$ arborally unsatisfied. So this case is not possible.

2. s lies above p_1 in $({}^{q}\square_{p})^{\circ}$.

   By the definition of $\textsc{Pair}(\cdot)$, q_1 is the first point above p_1. So, such a point s cannot exist.

3. $s \in ({}^{q}\square_{p_1})^{\circ}$ (see Fig. 18).

   Let us assume that s is the closest point to q in $({}^{q}\square_{p_1})^{\circ}$. Therefore, no point lies above s in $({}^{q}\square_{p_1})^{\circ}$. However, there exists a point above s that does not lie in $({}^{q}\square_{p_1})^{\circ}$. $\textsc{Up}(s)$ is one such candidate. Let t be the first point above s. Note that t can lie to the right or north-east of q (though in Fig. 18 it lies to the north-east of q).

   Note that q_1 hides t from p (as either q_1 lie to the right of t in ${}^{t}\square_{p}$ or $q_1 \in ({}^{t}\square_{p})^{\circ}$) and s is the first point below t such that ${}_{s}\swarrow\!\!\!\nearrow^{q_1}$. Then, by Lemma 4.6, ${}_{\textsc{Op}(s)}\swarrow\!\!\!\nearrow^{q_1}$. $\textsc{Op}(s)$ cannot lie in $({}^{q}\square_{p_1})^{\circ}$ as ${}^{q}\square_{p}$ is a good pair; this implies ${}_{\textsc{Op}(s)}\swarrow\!\!\!\nearrow^{q}$. In that case, q hides t from $\textsc{Op}(s)$ (as either q lies to the left of t in ${}_{\textsc{Op}(s)}\square^{t}$ or $q \in ({}_{\textsc{Op}(s)}\square^{t})^{\circ}$) and s is the first point below t such that ${}^{q}\nwarrow\!\!\!\searrow_{s}$. Again by Lemma 4.7, ${}^{q}\nwarrow\!\!\!\searrow_{\textsc{Op}(s)}$. This leads to a contradiction as we have already deduced that ${}_{\textsc{Op}(s)}\swarrow\!\!\!\nearrow^{q}$. So our assumption that $s \in ({}^{q}\square_{p_1})^{\circ}$ must be false.

$\square$

9.2 Interaction between Good Pairs

The definition of *independent sets* specifies how two rectangles can intersect each other. The set of good pairs need not be independent in general. However, there are constraints on the way good pairs interact. Specifically, we will show that if two good pairs intersect then the point associated with one of them cannot lie in the interior of the rectangle associated with the other pair:

Lemma 9.5. *Let* $\text{PAIR}(q_1) = {}^{s}\square_{q}$ *and* $\text{PAIR}(q_2) = {}^{p}\square_{q}$ *be two good pairs such that* ${}_{s}\swarrow\!\!\nearrow^{p}$. *Assume that the intersection between the two rectangles is of type* $\boxplus$. *Then* q_1 *cannot lie to the left of* q *in* ${}^{p}\square_{q}$ *except at the bottom-left corner of* ${}^{p}\square_{q}$.

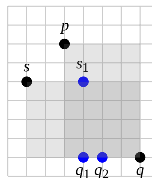


Figure 19: Illustration of the bad case in the proof of 9.5 when q_1 lies in to the left of q in ${}^{p}\square_{q}$ (but not at its bottom left corner).

Proof. The intersection of ${}^{p}\square_{q}$ and ${}^{s}\square_{q}$ is of type $\boxplus$. Let s_1 be the first point above q_1. If q_1 lies to the left of q in ${}^{p}\square_{q}$ (but not at the bottom-left corner of ${}^{p}\square_{q}$), then it lies to the left or right of q_2 in ${}^{p}\square_{q}$. This implies that s_1 lies in $({}^{p}\square_{q})^{\circ}$. However, Lemma 9.4 forbids such a situation.

$\square$

Lemma 9.6. *Let* $\text{PAIR}(q_1) = {}^{s}\square_{q}$ *and* $\text{PAIR}(r_1) = {}^{s}\square_{r}$ *be two good pairs such that* ${}_{r}\swarrow\!\!\nearrow^{q}$. *Assume that the intersection between the two rectangles is of type* $\boxplus$. *Then* $q_1 \notin ({}^{s}\square_{r})^{\circ}$ *and* q_1 *cannot be at the bottom-left corner of* ${}^{s}\square_{q}$.

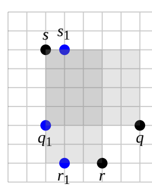


Figure 20: Illustration of the bad case in the proof of 9.6 when q_1 lies at the bottom-left corner of ${}^{s}\square_{q}$.

Proof. By Lemma 9.4, $q_1 \notin ({}^{s}\square_{r})^{\circ}$.

Assume for contradiction that q_1 lies at the bottom-left corner of ${}^{s}\square_{q}$, that is, it is the first point below s (as $\text{PAIR}(q_1) = (q,s)$). This implies that r_1 cannot lie below q_1 (as then $\text{PAIR}(r_1) \neq (r,s)$). So ${}^{q_1}\nwarrow\!\!\searrow_{r_1}$. Let s_1 be the first point above r_1. Since $\text{PAIR}(r_1) = (r,s)$, s_1 lies to the right of s. This implies that s_1 hides s from q (as s_1 lies to the right of s in ${}^{s}\square_{q}$) and q_1 is the first point below s. By Lemma 4.6, ${}_{\text{OP}(q_1)}\swarrow\!\!\nearrow^{s_1}$. But $\text{OP}(q_1) = q$ lies to the south-east of s_1. So, we arrive at a contradiction. So q_1 cannot lie at the bottom-left corner of ${}^{s}\square_{q}$

$\square$

The following two lemmas are also a direct consequence of Lemma 9.4.

Lemma 9.7. *Let* $\textsc{Pair}(q_1) = {}^{s}\square_{q}$ *and* $\textsc{Pair}(r_1) = {}^{p}\square_{r}$ *be two good pairs such that* ${}_{p}\swarrow\!\!\!\nearrow^{s}, {}_{r}\swarrow\!\!\!\nearrow^{q}$*. Assume that the intersection between the two rectangles is of type* ⧉*. Then* $q_1 \notin ({}^{p}\square_{r})^{\circ}$.

Lemma 9.8. *Let* $\textsc{Pair}(q_1) = {}^{s}\square_{q}$ *and* $\textsc{Pair}(r_1) = {}^{p}\square_{r}$ *be two good pairs such that* ${}_{s}\swarrow\!\!\!\nearrow^{p}, {}_{r}\swarrow\!\!\!\nearrow^{q}$*. Assume that the intersection between the two rectangles is of type* ⊞*. Then* $q_1 \notin ({}^{p}\square_{r})^{\circ}$.

9.3 Putting it together

Let $\textsc{Good}(\therefore)_{\boxbslash} := \{{}^{q}\square_{p} \mid {}^{q}\square_{p} \in \textsc{Good}(\therefore)\}$. Similarly we can define $\textsc{Good}(\therefore)_{\boxslash}$. Demaine et al. [5] proved that the size of an *independent set* of rectangles provides a lower bound on $|\textsc{Opt}(\mathscr{X})|$ (Claim 9.1). Instead of independent sets we have to argue about $\textsc{Good}(\therefore)$. Our argument would follow that of Demaine et al. at a high level, but with some important changes. We state three lemmas below which are adaptations of lemmas from [5] to $\textsc{Good}(\therefore)$.

Lemma 9.9. *(compare Lemma 4.4, [5]) Let q be the point in $\mathscr{X}$ with the maximum x-coordinate such that there exists a rectangle with q in $\textsc{Good}(\therefore)_{\boxbslash}$ as one of its diagonal point. Let ${}^{s}\square_{q} \in \textsc{Good}(\therefore)_{\boxbslash}$ be the widest rectangle with q as one of its diagonal point. Then we can find a vertical line ℓ passing through the interior of ${}^{s}\square_{q}$, such that inside ${}^{s}\square_{q}$, ℓ does not intersect the interior of any other rectangle in $\textsc{Good}(\therefore)_{\boxbslash} \setminus \{{}^{s}\square_{q}\}$.*

Proof. Let $\textsc{Pair}(q_1) = {}^{s}\square_{q}$. Let S denote the set of all rectangles in $\textsc{Good}(\therefore)_{\boxbslash} \setminus \{{}^{s}\square_{q}\}$ overlapping ${}^{s}\square_{q}$. Let us assume for contradiction that S spans the horizontal section of ${}^{s}\square_{q}$. Let ${}^{p}\square_{r}$ be a rectangle in S with $\textsc{Pair}(r_1) = {}^{p}\square_{r}$.

Note that any intersection between ${}^{s}\square_{q}$ and ${}^{p}\square_{r}$ can be of type ⧉, ⊞, ⊟ or ◫. This is due to the fact that any other intersection will either force two points from $\{p, q, r, s\}$ to be on the same horizontal or vertical line (which is forbidden as these points come from a permutation sequence), or it will force a diagonal point (in $\mathscr{X}$) of one rectangle to be in the interior of the other. Also, by the construction of ${}^{s}\square_{q}$, ${}^{p}\square_{r}$ is the left rectangle in the intersection type ⧉ and the narrower rectangle in intersection type ⊞, ⊟ or ◫.

The right edge of any rectangle in $\textsc{Good}(\therefore)_{\boxbslash}$ does not intersect even partially with the left edge of any other rectangle (as points in $\mathscr{X}$ come from a permutation sequence). So, if the rectangles in S span the horizontal section of ${}^{s}\square_{q}$, then each boundary point to the left of q in ${}^{s}\square_{q}$ (except maybe its bottom-left corner) lies either in the interior of a rectangle ${}^{p}\square_{r}$ (intersection type ⧉, ⊞, ◫) or to the left of q in ${}^{p}\square_{q}$ except at the bottom-left corner of ${}^{p}\square_{q}$ (intersection type ⊟ and $r = q$).

Then by Lemmas 9.5, 9.6, 9.7, and 9.8, point q_1 cannot lie to the left of q in ${}^{s}\square_{q}$, except possibly its bottom-left corner.

As rectangles of intersection type ⊞ and ⊟ are narrower than ${}^{s}\square_{q}$ (by the construction of ${}^{s}\square_{q}$), these rectangles cannot span horizontal section of ${}^{s}\square_{q}$. Since we assumed that rectangles in S span the horizontal section of ${}^{s}\square_{q}$, it implies that S contains a rectangle of type ⧉ or ◫. Using Lemmas 9.7 or 9.6 respectively, q_1 cannot even lie at the bottom-left corner of ${}^{s}\square_{q}$. This implies that q_1 does not lie to the left of q in ${}^{s}\square_{q}$. However, this contradicts Observation 6.4 that states if $\textsc{Pair}(q_1) = {}^{s}\square_{q}$, then q_1 lies to the left of q in ${}^{s}\square_{q}$. So we arrive at a contradiction, hence our assumption that "S spans the horizontal section of ${}^{s}\square_{q}$" must be false. So, we can find a vertical line ℓ through the interior ${}^{s}\square_{q}$, such that inside ${}^{s}\square_{q}$, ℓ does not intersect the interior of any other rectangle in $\textsc{Good}(\therefore)_{\boxbslash} \setminus \{{}^{s}\square_{q}\}$.

$\square$

Lemma 9.10. *(compare Lemma 4.3, [5]) Given* ${}^{p}\square_{q} \in \textsc{Good}(.\!:\!\cdot)_{\boxbslash}$ *and a vertical line* ℓ *at a non-integer x-coordinate intersecting* $({}^{p}\square_{q})^{\circ}$ *and a set of points* $\mathscr{Y}$ *such that each pair of points in* $\mathscr{X} \cup \mathscr{Y}$ *is arborally satisfied. We can find two points* $a, b \in (\mathscr{X} \cup \mathscr{Y})$ *in* ${}^{p}\square_{q}$ *such that* $a \leftrightarrow b$*,* a *is to the left of* ℓ*,* b *is to the right of* ℓ*, and there are no points in* $\mathscr{X} \cup \mathscr{Y}$ *on the horizontal segment connecting* a *to* b*.*

Proof. Let a and b in $(\mathscr{X} \cup \mathscr{Y})$ be two closest points in ${}^{p}\square_{q}$ such that a is the left of line ℓ and b is to the right of ℓ. Note that a exists (p is one such candidate). Similarly b exists (q is one such candidate). By construction, there are no points in ${}^{a}\square_{b}$ (or ${}_{a}\square^{b}$). If a and b do not lie on the same horizontal line, then ${}^{a}\square_{b}$ (or ${}_{a}\square^{b}$) is an arborally unsatisfied rectangle, contradicting our assumption that $(\mathscr{X} \cup \mathscr{Y})$ is an arborally satisfied set. $\square$

Lemma 9.11. *(compare Lemma 4.5, [5]) Given a point set* $\mathscr{X}$ *and another point set* $\mathscr{Y}$ *such that* $\mathscr{X} \cup \mathscr{Y}$ *is arborally satisfied, then* $|\mathscr{X} \cup \mathscr{Y}| \geqslant |\textsc{Good}(.\!:\!\cdot)_{\boxbslash}| + |\mathscr{X}|$.

Proof. This proof is essentially verbatim from [5]. The property of independent set of rectangles used in the proof of Lemma 4.5 in [5] is the existence of the line ℓ (proved in Lemma 4.4 there). Our Lemma 9.9 proves the existence of line ℓ for $\textsc{Good}(.\!:\!\cdot)_{\boxbslash}$. Given this, the proofs for independent set and for $\textsc{Good}(.\!:\!\cdot)_{\boxbslash}$ are the same.

We apply Lemma 9.9 to find a rectangle ${}^{s}\square_{q}$ in $\textsc{Good}(.\!:\!\cdot)_{\boxbslash}$ and a vertical line ℓ piercing ${}^{s}\square_{q}$ with the property that no other rectangle in $\textsc{Good}(.\!:\!\cdot)_{\boxbslash}$ intersects ℓ in the interior of ${}^{s}\square_{q}$. Then we apply Lemma 9.10 to find two points a, b horizontally adjacent in $\mathscr{X} \cup \mathscr{Y}$ and on opposite sides of ℓ in ${}^{s}\square_{q}$. We mark this pair (a, b) with rectangle ${}^{s}\square_{q}$. Then we remove ${}^{s}\square_{q}$ from $\textsc{Good}(.\!:\!\cdot)_{\boxbslash}$ and repeat the process, until there are no rectangles left in $\textsc{Good}(.\!:\!\cdot)_{\boxbslash}$. Whenever we remove a rectangle ${}^{s}\square_{q}$ from $\textsc{Good}(.\!:\!\cdot)_{\boxbslash}$, if a and b are not on the top or bottom sides of ${}^{s}\square_{q}$, then a and b do not simultaneously belong to any other rectangle in $\textsc{Good}(.\!:\!\cdot)_{\boxbslash} \setminus \{{}^{s}\square_{q}\}$, so they will never be marked again. On the other hand, if a and b are on the top (bottom) side of ${}^{s}\square_{q}$, then a and b are neither in the interior nor on the top (bottom) side of any other rectangle in $\textsc{Good}(.\!:\!\cdot)_{\boxbslash}$. Furthermore, since coordinates in $\mathscr{X}$ are distinct, the top side of no rectangle in $\textsc{Good}(.\!:\!\cdot)_{\boxbslash}$ coincides even partially with the bottom side of a rectangle in $\textsc{Good}(.\!:\!\cdot)_{\boxbslash}$. Thus, each pair of horizontally adjacent points in $\mathscr{X} \cup \mathscr{Y}$ can be marked at most once. Finally, by distinctness of y-coordinates in $\mathscr{X}$, at most one point in a pair can belong to $\mathscr{X}$. Therefore the number of points in $\mathscr{Y}$ is at least $|\textsc{Good}(.\!:\!\cdot)_{\boxbslash}|$, proving the lemma. $\square$

Similar to Lemma 9.11, we can also show that any arborally satisfied set $\mathscr{Y}$ satisfies $|\mathscr{X} \cup \mathscr{Y}| \geqslant |\textsc{Good}(.\!:\!\cdot)_{\boxslash}| + |\mathscr{X}|$. So $|\mathscr{X} \cup \mathscr{Y}| \geqslant \frac{|\textsc{Good}(.\!:\!\cdot)_{\boxbslash}| + |\textsc{Good}(.\!:\!\cdot)_{\boxslash}|}{2} + |\mathscr{X}| = \frac{|\textsc{Good}(.\!:\!\cdot)|}{2} + |\mathscr{X}|$. We have thus proved Theorem 9.2.

10 Upper bounding $|.\!:\!\cdot|$: Bad Pairs

In this section we prove

Lemma 10.1. $|\textsc{Bad}(.\!:\!\cdot)| \leqslant 10n(k-1)$.

For a point $p \in \mathscr{X}$, similar to $\mathrm{R}(\cdot\!.\!:)_{p}$, define $\mathrm{R}(\textsc{Bad}(.\!:\!\cdot))_{p} := \{\textsc{Pair}(p_1) \in \textsc{Bad}(.\!:\!\cdot) \mid p \leftrightarrow p_1 \text{ and } \textsc{Pair}(p_1) = (p, q), \text{ where } q \in \mathscr{X}\}$. Let $\textsc{Pair}(p_1) \in \mathrm{R}(\textsc{Bad}(.\!:\!\cdot))_{p}$. To upper bound $|\textsc{Bad}(.\!:\!\cdot)|$, we construct $\mathrm{F}(\cdot)$ that maps a point u (with $\textsc{Pair} \in \textsc{Bad}(.\!:\!\cdot)$) to a point v (with $\textsc{Pair}(v) \in \cdot\!.\!:$). $\mathrm{F}(\cdot)$ takes at most 4 points (with $\textsc{Pair}(\cdot) \in \textsc{Bad}(.\!:\!\cdot)$) to a point (with $\textsc{Pair}(\cdot) \in \cdot\!.\!:$), and we already proved $|\cdot\!.\!:| = O(nk)$. This would provide an upper bound on $|\textsc{Bad}(.\!:\!\cdot)|$. Unfortunately, $\mathrm{F}(\cdot)$ is a partial map, and does not map every point (with

PAIR$(\cdot) \in$ BAD(.:˙)). We show, by an argument similar to the one used for bounding $|$˙.:$|$, that the number of unmapped points is $O(nk)$. This gives our desired bound $|$BAD(.:˙)$| = O(nk)$.

The rest of this section is devoted to proving Lemma 10.1. In Sec. 10.1 we construct the map mentioned above; in Sections 10.2, 10.3 we provide upper bounds on the size of the sets of mapped and unmapped points; finally, we put these two bounds together to complete the proof.

10.1 Construction of F$(\cdot)$ and its properties

We begin by describing the setting used in the proof.

The setting used in the proof. Unless mentioned otherwise, we will assume in this section that TOPBLOCK$(B) = p$ and PAIR$(p_1) \in$ R(BAD(.:˙))$_p$. Let q_1 be the first point above p_1. Since GREEDY has put a point p_1 below q_1 while processing p, it found ${}_p\square^{q_1}$ arborally unsatisfied ($p \leftrightarrow p_1$ as PAIR$(p_1) \in$ R(BAD(.:˙))$_p$). This implies that there are no points in ${}_p\square^{q_1}$ except p, q_1 and p_1. If OP$(q_1) = q_1$, then PAIR$(p_1) \in$ GOOD(.:˙). So let us assume that OP$(q_1) = q$ and $q_1 \leftrightarrow q$ (since PAIR$(p_1) \in$ R(BAD(.:˙))$_p$ we do not have $q \leftrightarrow q_1$).

Since PAIR$(p_1) \in$ BAD(.:˙), there is another point in $\mathscr{X} \setminus \{p, q\}$ and in $({}_p\square^q)^\circ$. Let $r \in \mathscr{X}$ be a point in $({}_p\square^q)^\circ$ having the smallest x-distance from p. Let p'_1 be the next point to the right of p_1; that p'_1 exists is proven next:

Lemma 10.2. *There is a point p'_1 to the right of p in ${}_{p_1}\square^r$ but not at the bottom-right corner of ${}_{p_1}\square^r$.*

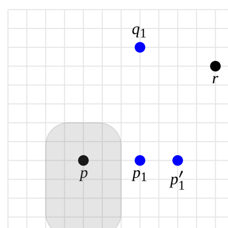


Figure 21: Lemma 10.2 states that p'_1 exists to the right of p_1 such that ${}_{p'_1}\swarrow\!\!\!\nearrow^{r}$

Proof. Consider the rectangle ${}^{q_1}\square_r$. By Lemma 4.4, a point r' lies either to the left or above r in ${}^{q_1}\square_r$. Since $r \in \mathscr{X}$, by Observation 4.1, GREEDY does not put any point above r. Also, since q_1 is the first point above p_1, r' cannot lie on the bottom-left endpoint of ${}^{q_1}\square_r$. So $r' \leftrightarrow r$ and ${}_{p_1}\swarrow\!\!\!\nearrow^{r'}$.

Now consider rectangle ${}_{p_1}\square^{r'}$. Again by Lemma 4.2, a point p'_1 must exist either to the right or above p_1 in ${}_{p_1}\square^{r'}$. Since q_1 is the first point above p_1 (and q_1 does not lie in ${}_{p_1}\square^{r'}$), p'_1 cannot lie above p_1 in ${}_{p_1}\square^{r'}$. So it must lie to the right of p_1 in ${}_{p_1}\square^{r'}$.

As $r' \leftrightarrow r$, we deduce that p'_1 cannot lie below r. $\square$

We are now ready to describe the construction of the partial map F$(\cdot)$. Its domain is $\{u \in \mathscr{G} \mid$ PAIR$(u) \in$ BAD(.:˙)$\}$ and its range is $\{v \in \mathscr{G} \mid$ PAIR$(v) \in$ ˙.:$\}$. F$(\cdot)$ is computed by the procedure in Fig. 22.

In the procedure for computing F(p_1), if we find that PAIR$(p'_1) \in$.:˙, then we look for s'_1, the first point above p'_1. We now prove some properties of s'_1; these will be used in Sec. 10.3. By Lemma 10.2, p'_1 cannot lie below r. So ${}_{p'_1}\swarrow\!\!\!\nearrow^{r}$ (see Fig. 23). Note that ${}_{s'_1}\swarrow\!\!\!\nearrow^{r}$ or ${}^{s'_1}\nwarrow\!\!\!\searrow_{r}$ or $s'_1 \leftrightarrow r$ (though in Fig. 23 we have shown the case ${}_{s'_1}\swarrow\!\!\!\nearrow^{r}$). Since GREEDY has put a point below s'_1 while processing p, hence ${}_p\square^{s'_1}$ is an arborally unsatisfied rectangle. So, ${}^{q_1}\nwarrow\!\!\!\searrow_{s'_1}$ as $p_1 \leftrightarrow p'_1$ and GREEDY marked these points due to unsatisfied rectangle ${}_p\square^{q_1}$ and ${}_p\square^{s'_1}$ respectively. We make some other observation regarding s'_1.

if $\textsc{Pair}(p_1') \in$ •.: **then**
  $\mathrm{F}(p_1) = p_1'$;
end
else
  Let s_1' be the first point above p_1';
  if $\textsc{Pair}(s_1') \in$ •.: **then**
    $\mathrm{F}(p_1) = s_1'$;
  end
  else
    $\mathrm{F}(p_1)$ is not defined;
  end
end

Figure 22: Procedure to compute $\mathrm{F}(p_1)$ mapping p_1 with $\textsc{Pair}(p_1) \in \mathrm{R}(\textsc{Bad}($.:•$))$ to a point with $\textsc{Pair}(\cdot) \in$ •.:. If $\textsc{Pair}(p_1) \in \mathrm{L}(\textsc{Bad}($.:•$))$, the procedure is symmetric.

Observation 10.3. *There are no points to the left of s_1' in ${}_p\square^{s_1'}$.*

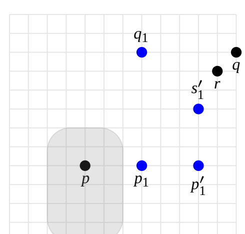
The above observation is a direct consequence of the fact that while processing p, GREEDY found ${}_p\square^{s_1'}$ arborally unsatisfied while processing p.

Observation 10.4. *There are no points in (1) $({}^{q_1}\square_{s_1'})^\circ$ and (2) below q_1 or to the left of s_1' in ${}^{q_1}\square_{s_1'}$.*

Again this is due to the facts that (1) p_1 and p_1' are two consecutive points to the right of p, (2) GREEDY found ${}_p\square^{q_1}$ and ${}_p\square^{s_1'}$ to be arborally unsatisfied while processing p, (3) q_1 is the first point above p_1.

Observation 10.5. *$s_1' \notin \mathscr{X}$ and $s_1' \leftrightarrow \textsc{Op}(s_1')$.*

Since r is an original point in ${}_p\square^q$ with the smallest x-distance to p, $s_1' \notin \mathscr{X}$. Together with our assumption that $\textsc{Pair}(p_1') \in$.:•, this implies $s_1' \leftrightarrow \textsc{Op}(s_1')$.



(i)

Figure 23: In $\mathrm{F}(p_1)$, s_1' is the first point above p_1'

10.2 Bounding the number of points mapped by $\mathrm{F}(\cdot)$

Let $\textsc{Pair}(p_i) \in \mathrm{R}(\textsc{Bad}($.:•$))_p$. For a $q_i \in \mathscr{G}$, if $\mathrm{F}(p_i) = q_i$, then $q_i \in \mathscr{G}$, and either q_i is the first point to the right of p_i or q_i is the first point above q_i' where q_i' is the first point to the right of p_i. Hence for $q_i \in \mathscr{G}$, there are at most two points p_i such that $\mathrm{F}(p_i) = q_i$. In other words, for each $q_i \in \mathscr{G}$, we have $|\{p_i \mid \mathrm{F}(p_i) = q_i$ and $\textsc{Pair}(q_i) \in$ •.:$\}| \leqslant 2$. So,

$$\sum_{p\in\mathscr{X}} |\{\textsc{Pair}(p_i) \in \mathrm{R}(\textsc{Bad}(\therefore))_p \mid p_i \in \mathrm{domain}(\mathrm{F})\}| = \sum_{q_i\in\mathscr{G}} |\{p_i \mid p_i \text{ s.t. } \mathrm{F}(p_i) = q_i \text{ and } \textsc{Pair}(q_i) \in \because\}|$$
$$\leqslant 2|\because|.$$

By symmetry,

$$\sum_{p\in\mathscr{X}} |\{\textsc{Pair}(p_i) \in \mathrm{L}(\textsc{Bad}(\therefore))_p \mid p_i \in \mathrm{domain}(\mathrm{F})\}| \leqslant 2|\because|.$$

Combining the above two inequalities we get

Lemma 10.6. $\sum_{p\in\mathscr{X}} |\{\textsc{Pair}(p_i) \in \textsc{Bad}(\therefore)_p \mid p_i \in \mathrm{domain}(\mathrm{F})\}| \leqslant 4|\because|$.

10.3 Bounding the number of points not mapped by F$(\cdot)$

We continue with the setting introduced in Sec. 10.1. Since $\mathrm{F}(\cdot)$ was not able to map p_1, the following must be true:

1. $\textsc{Pair}(p_1') \in \therefore$,
2. $\textsc{Pair}(s_1') \in \therefore$.

Since $\textsc{Pair}(p_1) \in \textsc{Bad}(\therefore)$, there exists another point $r \in \mathscr{X}\setminus\{p,q\}$ in ${}_p\square^q$.

Let t_1' be the first point above s_1'. So $\textsc{Pair}(s_1') = (\textsc{Op}(s_1'), \textsc{Op}(t_1'))$. Since $\textsc{Pair}(s_1') \in \therefore$ and $s_1' \leftrightarrow \textsc{Op}(s_1')$ (Observation 10.5), either $\textsc{Op}(t_1') \leftrightarrow t_1'$ or $\textsc{Op}(t_1') = t_1'$.

Observation 10.7. *Either* $\textsc{Op}(t_1') \leftrightarrow t_1'$ *or* $\textsc{Op}(t_1') = t_1'$.

We first make some claims on the position of t_1'.

Lemma 10.8. *Let t_1' be the first point above s_1'. Then* ${}^{q_1}\!\nwarrow\!\!\searrow_{t_1'}$.

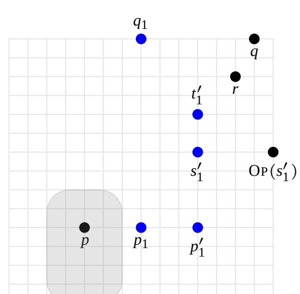


Figure 24: Lemma 10.8 shows that there exists a point t_1' such that (1) it is the first point above s_1' and (2) ${}^{q_1}\!\nwarrow\!\!\searrow_{t_1'}$

Proof. Consider the rectangle ${}^{q_1}\square_{s_1'}$. By Lemma 4.4, there exists a point t_1' to the left or above s_1' in ${}^{q_1}\square_{s_1'}$. However, by Observation 10.4, there is no point in (1) $({}^{q_1}\square_{s_1'})^\circ$ and (2) below q_1 or to the left of s_1' in ${}^{q_1}\square_{s_1'}$. This implies that $\underset{s_1'}{\overset{t_1'}{\updownarrow}}$ and $t_1' \in {}^{q_1}\square_{s_1'}$.

We now need to show that ${}^{q_1}\nwarrow\searrow_{t_1'}$. For contradiction, assume that $q_1 \leftrightarrow t_1'$. This implies that $\textsc{Op}(t_1') = q$ and $t_1' \leftrightarrow q$ (since $s_1'.x < r.x < q.x$ and $\underset{s_1'}{\overset{t_1'}{\updownarrow}}$). But we know by Observation 10.7 that $\textsc{Op}(t_1')$ cannot lie to the right of t_1'. Thus we have arrived at a contradiction. So our assumption that $q_1 \leftrightarrow t_1'$ must be false, and hence ${}^{q_1}\nwarrow\searrow_{t_1'}$.

$\square$

Our next claim is regarding the position of $\textsc{Op}(t_1')$.

Lemma 10.9. *Let t_1' be the first point above s_1'. Then,* $\textsc{Op}(t_1') \neq t_1'$, *and* ${}^{\textsc{Op}(t_1')}\nwarrow\searrow_{p}$, ${}_{\textsc{Op}(t_1')}\swarrow\nearrow^{q_1}$ *and* ${}^{\textsc{Op}(t_1')}\nwarrow\searrow_{s_1'}$.

Proof. By Observation 10.7, $\textsc{Op}(t_1') \leftrightarrow t_1'$ or $\textsc{Op}(t_1') = t_1'$. However, $\textsc{Op}(t_1') \neq t_1'$ because r is the the original point in ${}_{p}\square^{q}$ with the smallest x-distance from p. This implies that $\textsc{Op}(t_1') \leftrightarrow t_1'$.

We have to show that ${}^{\textsc{Op}(t_1')}\nwarrow\searrow_{p}$. If ${}_{p}\swarrow\nearrow^{\textsc{Op}(t_1')}$ (and $\textsc{Op}(t_1') \leftrightarrow t_1'$), it contradicts our assumption that r is the the original point in ${}_{p}\square^{q}$ with the smallest x-distance from p (recall that $r.x > p_1'.x = t_1'.x$ by Lemma 10.2). By Lemma 5.1, if $\textsc{TopBlock}(p) = B$, then there are no points in $\textsc{UpperBox}(B)$. Hence ${}^{\textsc{Op}(t_1')}\nwarrow\searrow_{p}$.

This together with $\underset{s_1'}{\overset{t_1'}{\updownarrow}}$ implies ${}^{\textsc{Op}(t_1')}\nwarrow\searrow_{s_1'}$. Also since ${}^{\textsc{Op}(t_1')}\nwarrow\searrow_{p}$, ${}_{p}\swarrow\nearrow^{q_1}$, ${}^{q_1}\nwarrow\searrow_{t_1'}$ (Lemma 10.8) and $\textsc{Op}(t_1') \leftrightarrow t'$, hence ${}_{\textsc{Op}(t_1')}\swarrow\nearrow^{q_1}$. $\square$

We say that a point p_1 is an *observable* point if $\mathrm{F}(p_1)$ is not defined.

Lemma 10.10. *Let $p_1 \leftrightarrow p_2$ be observable points such that* $\textsc{Pair}(p_1), \textsc{Pair}(p_2) \in \mathrm{R}(\textsc{Bad}(\cdot\!:\!\cdot))_p$ *for $p \in \mathscr{X}$. Assume that* $\textsc{Greedy}$ *put these points to arborally satisfy ${}_{p}\square^{q_1}, {}_{p}\square^{q_2}$, respectively, while processing point p. Then there is a point t such that ${}_{t}\swarrow\nearrow^{q_1}$ and ${}^{t}\nwarrow\searrow_{q_2}$.*

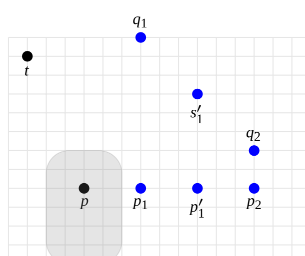


Figure 25: The setting of Lemma 10.10: p_1 and p_2 are two *observable* points. The above figure depicts the case when the next point to the right of p_1, $p_1' \neq p_2$.

Proof. One can check that $q_1.y < q_2.y$. Let p_1' be the first point to the right of p_1. And let s_1' be the first point above p_1' (see Fig. 25 for an illustration). Note that if $p_1' = p_2$, then $s_1' = q_2$. Else if $p_1 \leftrightarrow p_1' \leftrightarrow p_2$, then one can check that ${}^{s_1'}\nwarrow\searrow_{q_2}$. $\textsc{Greedy}$ put p_1' and p_2 due to the unsatisfied rectangles ${}_{p}\square^{s_1'}$ and ${}_{p}\square^{q_2}$,

respectively. By Lemma 10.9, there exists a t such that ${}_t\swarrow\!\!\!\!\nearrow^{q_1}$ and ${}^t\nwarrow\!\!\!\!\searrow_{s'_1}$. We can replace s'_1 with q_2 in the previous statement as either $s'_1 = q_2$ or ${}^{s'_1}\nwarrow\!\!\!\!\searrow_{q_2}$.

$\square$

Continuing the setting that was used in Lemma 10.10, let $\textsc{Pair}(p_i), \textsc{Pair}(p_{i+1}) \in \mathrm{R}(\textsc{Bad}(.\!:\!\cdot))_p$. If $\textsc{TopBlock}(p) = B$, then by Lemma 6.10, $\textsc{Op}(q_i)$ is in block B_i, where $B_i \in \textsc{Sibling}(B)$. Similarly, $\textsc{Op}(q_{i+1}) \in B_{i+1}$ such that $B_{i+1} \in \textsc{Sibling}(B)$. We now prove the following lemma that is similar to Lemma 7.3.

Lemma 10.11. *Let $p \in \mathscr{X}$ and $\textsc{TopBlock}(p) = B$. Let $p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_\ell$ be observable points such that $\textsc{Pair}(p_i) \in \mathrm{R}(\textsc{Bad}(.\!:\!\cdot))_p$ for $i \in [\ell]$. Assume that $\textsc{Greedy}$ puts these points due to arborally unsatisfied rectangles ${}_p\square^{q_1}, {}_p\square^{q_2}, \ldots, {}_p\square^{q_\ell}$, respectively, while processing point p. By Lemma 6.10, $\textsc{Op}(q_i) \in B_i$ such that $B_i \in \textsc{Sibling}(B)$. Then $B_i \neq B_j$ for $i \neq j$.*

Proof. Consider points $p_i \leftrightarrow p_j$. Since $\textsc{Pair}(p_i), \textsc{Pair}(p_j) \in \mathrm{R}(\textsc{Bad}(.\!:\!\cdot))_p$, we have $q_i \leftrightarrow \textsc{Op}(q_i)$ and $q_j \leftrightarrow \textsc{Op}(q_j)$ ($\textsc{Op}(q_i) \neq q_i$ as otherwise $\textsc{Pair}(p_j) \in \textsc{Good}(.\!:\!\cdot)$; similarly for $\textsc{Op}(q_j)$). By Lemma 10.10, there is a point t such that ${}_t\swarrow\!\!\!\!\nearrow^{q_i}$ and ${}^t\nwarrow\!\!\!\!\searrow_{q_j}$. This implies that ${}_t\swarrow\!\!\!\!\nearrow^{\textsc{Op}(q_i)}$ and ${}^t\nwarrow\!\!\!\!\searrow_{\textsc{Op}(q_j)}$. Moreover, by Lemma 10.9 we have ${}^t\nwarrow\!\!\!\!\searrow_p$. Hence $t.x < p.x < \textsc{Op}(q_i).x$, and $p \notin B_i$ (since $p \in B$ and B_i is the sibling of B). Hence, $t \notin B_i$ as all the points in block B_i should be contiguous in their keys.

Since ${}_t\swarrow\!\!\!\!\nearrow^{\textsc{Op}(q_i)}$ and ${}^t\nwarrow\!\!\!\!\searrow_{\textsc{Op}(q_j)}$, we also have $\textsc{Op}(q_i).y < t.y < \textsc{Op}(q_j).y$. Since $t \notin B_i$, $\textsc{Op}(q_i)$ and $\textsc{Op}(q_j)$ cannot be in the same block, as all the points in a block should be contiguous in time. Hence $B_i \neq B_j$. $\square$

10.4 Proof of Lemma 10.1

Since there are at most $k-1$ siblings of B, by Lemma 10.11 for each $p \in \mathscr{X}$ we have $|\{\textsc{Pair}(p_i) \in \mathrm{R}(\textsc{Bad}(.\!:\!\cdot))_p \mid p_i \text{ is an observable point}\}| \leqslant k-1$. By symmetry, $|\{\textsc{Pair}(p_i) \in \mathrm{L}(\textsc{Bad}(.\!:\!\cdot))_p \mid p_i \text{ is an observable point}\}| \leqslant k-1$ for $p \in \mathscr{X}$. So, $|\{\textsc{Pair}(p_i) \in \textsc{Bad}(.\!:\!\cdot)_p \mid p_i \text{ is an observable point}\}| \leqslant 2(k-1)$ for $p \in \mathscr{X}$. We are now ready to upper bound the number of points in $\textsc{Bad}(.\!:\!\cdot)$.

$$\begin{aligned}
|\textsc{Bad}(.\!:\!\cdot)| &= \sum_{p \in \mathscr{X}} \Big(|\{\textsc{Pair}(p_i) \in \textsc{Bad}(.\!:\!\cdot)_p \mid p_i \in \mathrm{domain(F)}\}| + |\{\textsc{Pair}(p_i) \in \textsc{Bad}(.\!:\!\cdot)_p \mid p_i \text{ is an observable point}\}| \Big) \\
&\leqslant 4|\cdot\!.\!:| + 2n(k-1).
\end{aligned}$$

Theorem 7.1 gives $|\cdot\!.\!:| \leqslant 2n(k-1)$. Hence $|\textsc{Bad}(.\!:\!\cdot)| \leqslant 10n(k-1)$, proving Lemma 10.1.

11 Improving the bound on the number of observable points in $.\!:\!\cdot$

Improving the bound in Lemma 10.1, we prove

Lemma 11.1. $|\textsc{Bad}(.\!:\!\cdot)| \leqslant O(n \log k)$.

We again define the notations used in Section 8. Let $B_1, B_2, \ldots, B_l$ be the children of B in $\textsc{TreeDecomposition}(\mathscr{X})$. Let $B_i, B_{i+1}, \ldots, B_{i+2m-1}$ be the consecutive $2m$ children of B. Let $\mathscr{B}_\ell = \{B_i, B_{i+1}, \ldots, B_{i+m-1}\}$ and $\mathscr{B}_r = \{B_{i+m}, B_{i+m+1}, \ldots, B_{i+2m-1}\}$. Let $N_{\textsc{Top}(B_j)}$ and $\mathrm{R}_{\textsc{Top}(B_j)}$ be the set of key-new and key-old elements added by $\textsc{Greedy}$ in $\textsc{Reg}(\mathscr{B}_r)$ while processing $\textsc{Top}(B_j)$. In Section 8.3, we proved that $\sum_{B_j \in \mathscr{B}_\ell} |R_{\textsc{Top}(B_j)}| \leqslant 12m$. Note that these points can be in $\cdot\!.\!:$ or $.\!:\!\cdot$. In this section, we need to bound the number of observable points in $N_{\textsc{Top}(B_j)}$. To this end, we will prove the following lemma which is similar to lemma 8.15.

Lemma 11.2. *Let $B_i, B_{i+1}, \dots, B_{i+2m-1}$ be the set of consecutive children blocks of B in* $\textsc{TreeDecomposition}(\mathscr{X})$*. The number of observable points added by* $\{\textsc{Top}(B_i), \textsc{Top}(B_{i+1}), \dots, \textsc{Top}(B_{i+m-1})\} \setminus \textsc{Top}(B)$ *in* $\textsc{Reg}(B_{i+m}, B_{i+m+1}, \dots, B_{i+2m-1})$ *is* $O(m)$*.*

Proof. Consider the points in $N_{\textsc{Top}(B_j)}$ where $B_j \in \mathscr{B}_\ell$. By Lemma 8.14, these points have $\textsc{Pair}(\cdot) \in .\!:\!\cdot$. Let $O_{\textsc{Top}(B)}$ be the set observable points in $N_{\textsc{Top}(B_j)}$. We will show that $\sum_{B_j \in \mathscr{B}_\ell} |O_{\textsc{Top}(B)}| \leqslant |\{B_k \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Left-Rel}(B_k)\}|$.

Consider any block $B_k \in \mathscr{B}_r$. By Corollary 8.11, only points in $\textsc{Left-Rel}(B_k)$ and $\textsc{Right-Rel}(B_k)$ can add key-new points in $\textsc{Reg}(B_k)$. Let us assume that $\textsc{Top}(B_j) \in \textsc{Left-Rel}(B_k)$ and assume that $\textsc{Greedy}$ adds two observable point $p_1 \leftrightarrow p_2$ in $\textsc{Reg}(B_k)$ while processing $\textsc{Top}(B_j)$ due to unsatisfied rectangle ${}_{\textsc{Top}(B_j)}\square^{q_1}, {}_{\textsc{Top}(B_j)}\square^{q_2}$ respectively. Since p_1 and p_2 are key-new points, q_1, q_2 lie in $\textsc{Box}(B_k)$. So $\textsc{Op}(q_1), \textsc{Op}(q_2) \in B_k$. However, this contradicts Lemma 10.11 which states that $\textsc{Op}(q_1)$ and $\textsc{Op}(q_2)$ are the elements of different siblings of B_k.

So, the number or observable points $|O_{\textsc{Top}(B_j)}|$ can be bounded by: $|\{B_k \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Left-Rel}(B_k)\}|$. And, $\sum_{B_j \in \mathscr{B}_\ell} |O_{\textsc{Top}(B)}| \leqslant \sum_{B_j \in \mathscr{B}_\ell} |\{B_k \in \mathscr{B}_r | \textsc{Top}(B_j) \in \textsc{Left-Rel}(B_k)\}|$. Since there are at most 2 left-relative of any block, the above inequality can be written as: $\sum_{B_j \in \mathscr{B}_\ell} |O_{\textsc{Top}(B)}| \leqslant \sum_{B_k \in \mathscr{B}_r} 2 = 2m$.

Since, points in $R_{\textsc{Top}(\cdot)}$ may also be observable, the total number of observable points added in $\textsc{Reg}(\mathscr{B}_r)$ while processing points in $\{\textsc{Top}(B_i), \textsc{Top}(B_{i+1}), \dots, \textsc{Top}(B_{i+m-1})\} \setminus \textsc{Top}(B) = \sum_{B_j \in \mathscr{B}_\ell} (R_{\textsc{Top}(B_j)} + O_{\textsc{Top}(B_j)}) \leqslant 12m + 2m = 14m$. □

Consider a block B in $\textsc{TreeDecomposition}(\mathscr{X})$ having children $B_1, B_2, \dots, B_\ell (\ell \leqslant k)$. Let $Y(B) := T(B_1, B_2, \dots, B_\ell)$ be the total number of observable points added by $\textsc{Greedy}$ while processing points in $\{\textsc{Top}(B_1), \textsc{Top}(B_2), \dots, \textsc{Top}(B_\ell)\} \setminus \textsc{Top}(B)$ in $\textsc{Reg}(B_1, B_2, \dots, B_\ell)$. Then using lemma 11.2 and its symmetric version, we can calculate $Y(B)$ as follows: $Y(B) = T(B_1, B_2, \dots, B_\ell) = T(B_1, B_2, \dots, B_{\ell/2}) + T(B_{\ell/2+1}, B_{\ell/2+2}, \dots, B_\ell)$ + 14ℓ. This would imply that $Y(B) = T(B_1, B_2, \dots, B_\ell) \leqslant 14\ell \log \ell$.

We would charge these $O(\ell \log \ell)$ points to the following $\ell - 1$ original points: $\{\textsc{Top}(B_1), \textsc{Top}(B_2), \dots, \textsc{Top}(B_\ell)\} \setminus \textsc{Top}(B)$. That is, each top point of children of block B except one gets $15 \log \ell$ charge. Similar to Theorem 8.1, we can bound the number of observable points.

Theorem 11.3. $\sum_{p \in \mathscr{X}} |\{\textsc{Pair}(p_i) \in \textsc{Bad}(.\!:\!\cdot)_p \mid p_i \textit{ is an observable point}\}| = 16 n \log k$

Proof. Let $\textsc{TopBlock}(p) = B'$ and $\textsc{Parent}(B') = B$. By Lemma 8.4, $\textsc{Greedy}$ can put at most two point below $\textsc{Left}(B)$ and $\textsc{Right}(B)$. And by the analysis above, the amortized number of observable points added by $\textsc{Greedy}$ in $\textsc{Box}(B)$ while processing p is $15 \log \ell = 15 \log k$ where ℓ is the number of children of B. So, amortized number of points added by $p = 2 + 15 \log k$. So $\sum_{p \in \mathscr{X}} |\{\textsc{Pair}(p_i) \in \textsc{Bad}(.\!:\!\cdot)_p \mid p_i \text{ is an observable point}\}| \leqslant 16 n \log k$. □

11.1 Proof of Lemma 11.1

We are now ready to upper bound the number of points in $\textsc{Bad}(.\!:\!\cdot)$.

$$\begin{aligned}|\textsc{Bad}(.\!:\!\cdot)| &= \sum_{p \in \mathscr{X}} \Big(|\{\textsc{Pair}(p_i) \in \textsc{Bad}(.\!:\!\cdot)_p \mid p_i \in \text{domain(F)}\}| + |\{\textsc{Pair}(p_i) \in \textsc{Bad}(.\!:\!\cdot)_p \mid p_i \text{ is an observable point}\}| \Big) \\ &\leqslant 4|\cdot\!.\!:| + 16 n \log k.\end{aligned}$$

Theorem 7.1 gives $|\cdot\!.\!:| \leqslant 14 n \log k$. Hence $|\textsc{Bad}(.\!:\!\cdot)| \leqslant 80 n \log k$, proving Lemma 11.1.

12 Proof of the main result

With all the ingredients at hand, the proof of the main result Theorem 1.1 is now short:

$$
\begin{aligned}
|\mathscr{G}| &\leqslant 2|\textsc{Pair}(\mathscr{G})| && \text{(Corollary 6.6)}\\
&= 2(|\;| + |\;|) && \text{(Lemma 6.8)}\\
&= 2(|\;| + |\textsc{Bad}(\;)| + |\textsc{Good}(\;)|) &&\\
&\leqslant 188n\log k + 2|\textsc{Good}(\;)| && \text{(Theorem 7.1 and Lemma 10.1)}\\
&\leqslant 188n\log k + 4|\mathscr{X} \cup \textsc{Opt}(\mathscr{X})|. && \text{(Theorem 9.2)}
\end{aligned}
$$

Using Corollary 1.10 in [2], namely $|\mathscr{X} \cup \textsc{Opt}(\mathscr{X})| = O(n\log k)$ for k-decomposable sequences, we get $|\mathscr{G}| = O(n\log k)$, which immediately gives Theorem 1.1.

References

[1] Prosenjit Bose, Karim Douïeb, Vida Dujmovic, and Rolf Fagerberg. An $O(\log\log n)$-competitive binary search tree with optimal worst-case access times. In *Algorithm Theory - SWAT 2010*, pages 38–49, 2010.

[2] Parinya Chalermsook, Mayank Goswami, László Kozma, Kurt Mehlhorn, and Thatchaphol Saranurak. Pattern-avoiding access in binary search trees. In *FOCS*, volume abs/1507.06953, 2015.

[3] Richard Cole. On the Dynamic Finger Conjecture for Splay Trees. Part II: The Proof. *SIAM J. Comput.*, 30(1):44–85, 2000.

[4] Richard Cole, Bud Mishra, Jeanette P. Schmidt, and Alan Siegel. On the Dynamic Finger Conjecture for Splay Trees. Part I: Splay Sorting $\log n$-Block Sequences. *SIAM J. Comput.*, 30(1):1–43, 2000.

[5] Erik D. Demaine, Dion Harmon, John Iacono, Daniel M. Kane, and Mihai Patrascu. The geometry of binary search trees. In *SODA*, pages 496–505, 2009.

[6] Erik D. Demaine, Dion Harmon, John Iacono, and Mihai Patrascu. Dynamic Optimality - Almost. *SIAM J. Comput.*, 37(1):240–251, 2007.

[7] Jonathan Derryberry, Daniel Dominic Sleator, and Chengwen Chris Wang. A Lower Bound Framework for Binary Search Trees with Rotations. *Tech Report, CMU*.

[8] Kyle Fox. Upper bounds for maximally greedy binary search trees. In *Algorithms and Data Structures - 12th International Symposium, WADS*, pages 411–422, 2011.

[9] John Iacono. In pursuit of the dynamic optimality conjecture. In *Space-Efficient Data Structures, Streams, and Algorithms - Papers in Honor of J. Ian Munro on the Occasion of His 66th Birthday*, pages 236–250, 2013.

[10] Joan M. Lucas. Canonical forms for competitive binary search tree algorithms. *Tech. Rep. DCS-TR-250, Rutgers University*, 1988.

[11] J. Ian Munro. On the Competitiveness of Linear Search. In *Algorithms - ESA 2000, 8th Annual European Symposium, Saarbrücken, Germany, September 5-8, 2000, Proceedings*, pages 338–345, 2000.

[12] Seth Pettie. Splay trees, davenport-schinzel sequences, and the deque conjecture. In *SODA*, pages 1115–1124, 2008.

[13] Seth Pettie. Applications of forbidden 0-1 matrices to search tree and path compression-based data structures. In *SODA*, pages 1457–1467, 2010.

[14] Daniel Dominic Sleator and Robert Endre Tarjan. Self-Adjusting Binary Search Trees. *J. ACM*, 32(3):652–686, 1985.

[15] Chengwen Chris Wang, Jonathan Derryberry, and Daniel Dominic Sleator. $O(\log\log n)$-competitive dynamic binary search trees. In *SODA*, pages 374–383, 2006.

[16] Robert E. Wilber. Lower Bounds for Accessing Binary Search Trees with Rotations. *SIAM J. Comput.*, 18(1):56–67, 1989.